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Hindawi Publishing Corporation
http://www.hindawi.com

Hindawi Publishing Corporation
410 Park Avenue, 15th Floor, #287 pmb, New York, NY 10022, USA
Nasr City Free Zone, Cairo 11816, Egypt
Fax: +1-866-HINDAWI (USA Toll-Free)

ISBN 978-977-454-099-8

Dedication

Dedicated to My Teacher
B. A. Shcherbakov

Contents

Preface

Nonlocal problems concerning the conditions of the existence of different classes of solutions play an important role in the qualitative theory of differential equations. Here belong the problem of boundedness, periodicity, almost periodicity, stability in the sense of Poisson, and the problem of the existence of limit regimes of different types, convergence, dissipativity, and so on. The present work belongs to this direction and is dedicated to the study of asymptotically stable in the sense of Poisson (in particular, asymptotically almost periodic) motions of dynamical systems and solutions of differential equations.

There is series of works of known authors dedicated to the problem of asymptotically stability in the sense of Poisson.

First the notion of asymptotically almost periodicity of functions it was introduced and studied in the works of Fréchet [1, 2]. Later these results were generalized for asymptotically almost periodic sequences in the works of Fan [3] and Precupanu [4] and for abstract asymptotically almost periodic functions in the works of Ararktsyan [5] and Precupanu [4] and also Khaled [6], Cioranescu [7], Dontvi [8, 9], Mambriani and Manfredi [10], and Manfredi [11, 12], Marchi [13, 14], Ruess and Summers [15–18], Seifert [19], Vesentini [20] and see also the bibliography therein.

Other series of works Antonishin [21], Arendt and Batty [22], Barac [23], Barbalat [24], Khaled [6], Bogdanowicz [25], Buşe [26], Casarino [27], Chen and Matano [28], Chepyzhov and Vishik [29], Coppel [30], Corduneanu [31], Dontvi [8, 9, 32], Draghichi [33], Fink [34], Gerko [35], Gheorghiu [36], Grimmer [37], Guryanov [38], Nacer [39], Henriquez [40], Hino, Murakami and Yoshizawa [41], Hino and Murakami [42, 43], Jordan and Wheeler [44], Jordan, Madych and Wheeler [45], Yao, Zhang, and Wu [46], Lovicar [47], Manfredi [48–51], Miller [52, 53], Muntean [54, 55], Puljaev and Caljuk [56, 57], Risito [58], Ruess and Phong [59], Sandberg and Zyl [60], Seifert [19, 61–63], Staffans [64], Shen and Yi [65], Taam [66], Tudor [67], Utz and Waltman [68], Vuillermot [69], Yamaguchi [70], Yamaguchi and Nishihara [71], Yuan [72], Yoshizawa [73], Zaidman [74, 75], Zhang [76] (see also the bibliography therein) is dedicated to the problem of asymptotically almost periodicity of solutions of differential (both ODEs and PDEs), functional-differential and integral equations.

At last, in the works of Khaled [6], Bhatia [77], Bhatia and Chow [78], Bronshteyn and Černii [79], Gerko [35, 80], Hino and Murakami [42], Millionshchikov [81, 82], Nemytskii [83, 84], Ruess and Summers [85], Seifert [19, 63], Sibirskii [86] and others they are studied motions of dynamical systems that are close by their properties to asymptotically almost periodic ones.

From the above said it follows that the problem of asymptotically stability in the sense of Poisson was studied earlier mainly for asymptotically periodicity and asymptotically almost periodicity of motions of dynamical systems and solutions of differential and integral equations. In this domain there were obtained important results, however the problem was not studied thoroughly.

In the present work there is studied the general problem the asymptotically Poisson stability of motions of dynamical systems and solutions of differential equations.

From the point of view of applications motions of dynamical systems are naturally divided on transitional (nonstabilized) and stabilized. By transitional we mean the motions that under unlimited increasement of time asymptotically approach to some established motion, that is, a motion that possesses some property of recurrence and stability.

When we try to define a nonstabilized motion exactly we come to the notion of the asymptotically stability in the sense of Poisson motion. Such motions are of interest for applications and are met, for instance, in systems possessing stable oscillatory regime (e.g., under the phenomenon of convergence).

The used in the present work method of study is based on the results of the topological theory of dynamical systems and can be applied for various types of asymptotically stability in the sense of Poisson. The idea of applying the methods of the theory of dynamical system while studying nonautonomous differential equations in itself is not new. It is successfully applied for solution of different problems in the theory of linear and nonlinear nonautonomous differential equations. First such approach to nonautonomous differential equations was applied in the works of Deysach and Sell [87], Miller [52], Millionshchikov [81], Seifert [61], Sell [88, 89], Shcherbakov [90–92], later in the works of Bronshteyn [93], Zhikov [94] and many other authors. It consists in natural binding with every nonautonomous equation a pair of dynamical systems and a homomorphism of the first onto the second. In this case, roughly saying, we enclose the information about the right-hand side of the equation in one dynamical system and the information about the solutions of this equations we put in the second system.

The offered work consists of five chapters.

In first chapter there are introduced and studied asymptotically almost periodic motions of abstract dynamical systems. There are given various criterions of asymptotically periodicity and asymptotically almost periodicity of motions. Applying the obtained results to the dynamical system of shifts (Bebutov system) in the space of continuous functions, we get the known results of Fréchet [1, 2]. We also consider the system of shifts in the space of locally summable functions, S^p asymptotically almost periodic functions (asymptotically almost periodic function in the sense of Stepanov) and establish series of their most important properties.

The second chapter is dedicated to asymptotically almost periodic solutions of operator equations. The notion of compatibility with respect to the character of recurrence of motions introduced by Shcherbakov [92] for the stable in the sense of Poisson motions is generalized on asymptotically stable by Poisson motions. Namely, the notion of compatibility of motions with respect to the character of recurrence in limit is introduced. There is established that compatible with respect to the recurrence in limit motions belong to the same classes of asymptotically stability by Poisson. There are obtained various tests of asymptotically stability in the sense of Poisson of solutions of operator equations. There are studied homoclinic and heteroclinic trajectories of dynamical systems. There are established tests of convergence of asymptotically almost periodic systems.

In Chapter 3 there are studied asymptotically almost periodic solutions of differential equations. There are given tests of the existence of compatible in limit solutions of different classes of differential equations. For asymptotically almost periodic systems there is established an analogue of the second theorem of Bogolyubov [95] (the averaging principle on the semiaxis). There are studied bilaterally asymptotically almost periodic solutions of some classes of equations and asymptotically almost periodic systems with convergence.

The fourth chapter is dedicated to asymptotically almost periodic linear systems with generalized perturbations. There are studied bounded on the semiaxis and asymptotically almost periodic distributions. There are given necessary and sufficient conditions of solvability of linear asymptotically almost periodic equations in the space of asymptotically almost periodic distributions. There are given tests of the existence of weakly asymptotically almost periodic solutions of linear and quasilinear differential equations.

In the fifth chapter there are studied asymptotically almost periodic solutions of functionally differential equations both with finite and infinite delay. There are established tests of the existence of asymptotically almost periodic solutions of integral equations of Volterra. There are given the conditions of convergence of some evolutionary equations with asymptotically almost periodic coefficients.

The given in the work results belong mainly to the author (besides Chapter 4, in which there are presented the results of Dontvi Isaac (excepting Section 4.6)) and are published in his works [8, 9, 32, 96, 97] and a part of these results (Sections 2.5–2.7, 3.5–3.8, 5.4, and 5.6-5.7) is published for the first time here.

For the best dividing of the material and outlining of the places that are of importance, we emphasize not only lemmas and theorems but as well many corollaries, remarks and examples.

The author hopes that the offered book will be useful both for both experts and young researchers who are interested in dynamical systems and their applications.

The reader needs no deep knowledge of special branches of mathematics. Despite this, however, it will be helpful for the reader to known the fundamentals of the qualitative theory of differential equations.

Not having a usual practice of English, the quality of the English of this book is certainly affected. The reader may excuse this fact.

Acknowledgments

The research described in this publication, in part, was possible due to Award No. MM1-3016 of the Moldovan Research and Development Association (MRDA) and the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).

October, 2008

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Notation

$\forall$	for every;
$\exists$	exists;
$:=$	equals (coincides) by definition;
0	zero, and also zero element of any additive group (semigroup);
$\mathbb{N}$	is the set of all natural numbers;
$\mathbb{Z}$	is the set of all integer numbers;
$\mathbb{Q}$	is the set of all rational numbers;
$\mathbb{R}$	is the set of all real numbers;
$\mathbb{C}$	is the set of all complex numbers;
$\mathbb{S}$	is one of the sets $\mathbb{R}$ or $\mathbb{Z}$;
$\mathbb{S}_+(\mathbb{S}_-)$	is the set of all nonnegative (nonpositive) numbers from $\mathbb{S}$;
$X \times Y$	is the Decart product of two sets;
M^n	is the direct product of n copies of the set M;
E^n	is the real or complex n-dimensional Euclidian space;
$\{x_n\}$	is a sequence;
$x \in X$	is an element of the set X;
∂X	is the boundary of the set X;
$X \subseteq Y$	the set X is a part of the set Y or coincides with it;
$X \bigcup Y$	is the union of the sets X and Y;
$X \setminus Y$	is the complement of the set Y in X;
$X \bigcap Y$	is the intersection of the sets X and Y;
$\varnothing$	the empty set;
(X, ρ)	is a full metric space with the metric ρ;
$\overline{M}$	is the closure of the set M;
f^{-1}	is the mapping inverse to f;
$f(M)$	is the image of the set $M \subseteq X$ in the mapping $f : X \to Y$, that is, $\{y \in Y : y = f(x), x \in M\}$;
$f \circ g$	is the composition of the mappings f and g, that is, $(f \circ g)(x) = f(g(x))$;
$f\vert_M$	is the restriction of the mapping f on the set M;
$f(\cdot, x)$	is the partial mapping defined by the function f when the second argument is x;
Id_X	is the the identity mapping of X into X;
$Im(f)$	is the range of values of the function f;
$D(f)$	is the domain of definition of the function f;
$\vert x\vert$ or $\Vert x\Vert$	is the norm of the element x;
(x, y)	an ordered pair;
$C(X, Y)$	is the set of all continuous mappings of the space X in the space Y endowed with the compact-open topology;

$C^k(U, M)$	is the set of all k times continuously differentiable mappings of the manifold U into the manifold M;
$f : X \to Y$	is a mapping of X into Y;
$B(M, \varepsilon)$	is an open ε-neighborhood of the set M in the metric space X;
$B[M, \varepsilon]$	is a closed ε-neighborhood of the set M in the metric space X;
$\{x, y, \dots, z\}$	is a set consisting of $x, y, \dots, z$;
$\overline{1, n}$	is a set consisting of $1, 2, \dots, n$;
$\{x \in X \mid \mathfrak{R}(x)\}$	is the set of all the elements from X possessing the property $\mathfrak{R}$;
$f^{-1}(M)$	is the preimage of the set $M \subseteq Y$ in the mapping $f : X \to Y$, that is, $\{x \in X : f(x) \in M\}$;
$F(t, \cdot) := f^t$	is the partial mapping given by the function f when the second argument is t;
$\rho(\xi, \eta)$	is a distance in the metric space X;
$\lim\limits_{n \to +\infty} x_n$	is the limit of a sequence;
$\varepsilon_k \downarrow 0$	is a monotonically decreasing to 0 sequence;
$\lim\limits_{x \to a} f(x)$	is the limit of mapping f as $x \to a$;
$\bigcup\{M_\lambda : \lambda \in \Lambda\}$	is the union of the family of sets $\{M_\lambda\}_\lambda \in \Lambda$;
$\bigcap\{M_\lambda : \lambda \in \Lambda\}$	is the intersection of the family of sets $\{M_\lambda\}_\lambda \in \Lambda$;
$(\mathcal{H}, \langle \cdot, \cdot \rangle)$	is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$;
$C(X)$	is the set of all compacts from X;
$(X, \mathbb{T}, \pi)$	is a dynamical system;
(X, P)	is the cascade generated by positive powers of P;
$\omega_x(\alpha_x)$	is the $\omega(\alpha)$-limit set of the point x;
α_{φ_x}	is the α-limit set of the entire trajectory $\varphi_x \in \Phi_x$;
Φ_x	is the set of all entire trajectories of the dynamical system $(X, \mathbb{T}, \pi)$ issuing from the point x as $t = 0$;
$W^s(M)$	is the stable manifold (the domain of attraction) of the set M;
M is st. L^+	the set M is stable in the sense of Lagrange in positive direction;
$D_x^+ \ (J_x^+)$	is a positive (positive limit) prolongation of the point x;
xt	is the position of the point x in the moment of time t;
pr_i	is the projection of $X_1 \times X_2$ onto the component of X_i with the index $i \ (i = 1, 2)$;
$D^+(M)$	is a positive prolongation of the set M;
$J^+(M)$	is a positive limit prolongation of the set M;
Σ_x^+	is a positive semitrajectory of the point x;
$\Sigma^+(M)$	is a positive semitrajectory of the set M;
$H^+(x)$	is a closure of the positive semitrajectory of the point x;
Σ_x	is the trajectory of the point x;
$H(x)$	is the closure of the trajectory of the point x;
Ω	is the closure of the union of all ω-limit points of $(X, \mathbb{T}, \pi)$;
$\mathfrak{M}_x$	is the set of all directing sequences of the point x;
$\mathfrak{N}_x$	is the set of all proper sequences of the point x;
$\mathfrak{L}_x$	is the set of all sequences $\{t_n\} \in \mathfrak{M}_x$ satisfying the condition $\lvert t_n \rvert \to +\infty$;
$\beta(A, B)$	is the semideviation of the set A from the set B $(A, B \in 2^X)$;

$D^m_{L^1}(\mathbb{R}_+)$	are the spaces of functions $\varphi : \mathbb{R}_+ \to E^n$ possessing $m-1$ usual derivatives, and $D^{m-1}\varphi$ is absolutely continuous and $D^j\varphi \in L^1(\mathbb{R}_+)$, $0 \le j \le m$;
$D^\infty_{L_1}(\mathbb{R}_+)$	is the space of infinitely differentiable functions, all the derivatives of which belong to $L^1(\mathbb{R}_+)$.
$\mathcal{D}(Q)$	is the space of infinitely differentiable functions $\varphi : Q \to E^n$ with the compact carrier;
$\mathcal{D}^m(Q)$	the space of functions $\varphi : Q \to E^n$ with m continuous derivatives and the compact carrier;
$C^m(Q)$	is the array of all the functions $\varphi : Q \to E^n$ having continuous derivatives up to order m inclusively;
$C^m(\overline{Q})$	family of all the functions φ from $C^m(Q)$, for which all derivatives $D^m\varphi$ admit a continuous prolongation onto $\overline{Q}$;
$\mathcal{D}'(Q)$	is the space adjoint to $\mathcal{D}(Q)$;
$\beta'^m(\mathbb{R}_+)$	is the space adjoint to $\mathcal{D}^m_{L^1}(\mathbb{R}_+)$ $(0 \le m < +\infty)$;
$\beta^m(\mathbb{R}_+)$	is the set of all multiplicators in $\beta'^m(\mathbb{R}_+)$;
$\beta^m_{app}(\mathbb{R}_+)$	is the space of all asymptotically almost periodic functions $\varphi \in \beta^m(\mathbb{R}_+)$;
$\beta^\infty_{app}(\mathbb{R}_+)$	is the space of all the functions that are asymptotically almost periodic together with their derivatives;
$\beta'^m_{app}(\mathbb{R}_+)$	is the space of distributions $f \in \beta'^m(\mathbb{R}_+)$, the shifts of which $\{\tau_n f \mid h \in \mathbb{R}_+\}$ form a relatively compact set in $\beta'^m(\mathbb{R}_+)$;
$AP(\mathbb{R}_+)$	denotes the set of all asymptotically almost periodic functions from $C(\mathbb{R}_+, E^n)$;
AP^m	is the set of all m times continuously differentiable functions from $C(\mathbb{R}_+, E^n)$ that are asymptotically almost periodic together with their derivatives up to the order m inclusively;
$(X', \mathbb{R}, \pi')$	is the dynamical system adjoint to $(X, \mathbb{R}, \pi)$;
$D = D(\mathbb{R})$	is the space of all finite continuously differentiable functions $\varphi : \mathbb{R} \to \mathbb{R}^n$;
X'	is the space of all linear continuous functionals on X;
$C_b(\mathbb{T}, E^n)$	is the Banach space of all continuous and bounded functions $f : \mathbb{T} \to E^n$ with the sup-norm;
$(C^*_b(\mathbb{T}, E^n))^n$	is the space adjoint to $(C_b(\mathbb{T}, E^n))^n$;
$U(t, A)$	is the operator of Cauchy;
$G_A(t, \tau)$	is the function of Green;
$D(A)$	is the domain of definition of the operator A.

1 Asymptotically Almost Periodic Motions

1.1. Some Notions and Denotations

Let us give some notions and denotations used in the theory of dynamical systems [86, 92, 93, 98–100] which we will apply in the present book.

Let X be a topological space, $\mathbb{R}$ ($\mathbb{Z}$) a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) a semigroup of nonnegative real (integer) numbers, $\mathbb{S}$ one of subsets of $\mathbb{R}$ or $\mathbb{Z}$, and $\mathbb{T} \subseteq \mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$, where $\mathbb{S}_+ = \{s \mid s \in \mathbb{S},\ s \geq 0\}$ is a semigroup of additive group $\mathbb{S}$).

Definition 1.1. The triplet $(X, \mathbb{T}, \pi)$, where $\pi : X \times \mathbb{T} \to X$ is a continuous mapping satisfying the following conditions:

$$\pi(0, x) = x \quad (x \in X,\ 0 \in \mathbb{T}), \tag{1.1}$$

$$\pi\big(\tau, \pi(t, x)\big) = \pi(t + \tau, x) \quad (x \in X;\ t, \tau \in \mathbb{T}) \tag{1.2}$$

are called a dynamical system. In that case if $\mathbb{T} = \mathbb{R}_+$ ($\mathbb{R}$) or $\mathbb{Z}_+$ ($\mathbb{Z}$) then the system $(X, \mathbb{T}, \pi)$ is called a semigroup (group) dynamical system. If $\mathbb{T} = \mathbb{R}_+$ ($\mathbb{R}$) the dynamical system is called flow and if $\mathbb{T} \subseteq \mathbb{Z}$ then $(X, \mathbb{T}, \pi)$ is called cascade.

To be short we will write instead of $\pi(t, x)$ just xt or $\pi^t x$. Further, as a rule, X will be a complete metric space with the metric ρ.

Definition 1.2. The function $\pi(\cdot, x) : \mathbb{T} \to X$ with fixed $x \in X$ is called motion of the point x and the set $\Sigma_x := \pi(\mathbb{T}, x)$ is called trajectory of this motion or of the point x.

Let $\mathbb{T} \subseteq \mathbb{T}'$ ($\mathbb{T}'$ is a subsemigroup from $\mathbb{S}$).

Definition 1.3. The motion $\pi(\cdot, x) : \mathbb{T} \to X$ is called extendible on $\mathbb{T}'$ if there exists a continuous mapping $\gamma : \mathbb{T}' \mapsto X$ such that

(1) $\gamma|_{\mathbb{T}} = \pi(\cdot, x)$;
(2) $\pi(t, \gamma(s)) = \gamma(t + s)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{T}'$.

Denote by $\Phi_x := \{(\gamma, \mathbb{T}') : \gamma$ is a extension on $\mathbb{T}'$ of motion $\pi(\cdot, x)\}$.

Definition 1.4. If for any point $x \in X$ and $(\gamma_1, \mathbb{T}'), (\gamma_2, \mathbb{T}'') \in \Phi_x$ from the equality $\gamma_1(t_0) = \gamma(t_0)$ it follows $\gamma_1(t) = \gamma_2(t)$ for all $t \in \mathbb{T}' \bigcap \mathbb{T}''$, then $(X, \mathbb{T}, \pi)$ is said to be a semigroup dynamical system with uniqueness.

Remark 1.5. We will suppose that any semigroup dynamical system, considering in this book, possesses the property of uniqueness.

Definition 1.6. A nonempty set $\mathbf{M} \subseteq X$ is called positively invariant (resp., negatively invariant, invariant) if $\pi(t, \mathbf{M}) \subseteq \mathbf{M}$ (resp., $\pi(t, \mathbf{M}) \supseteq \mathbf{M}$, $\pi(t, \mathbf{M}) = \mathbf{M}$) for all $t \in \mathbb{T}$.

Definition 1.7. A closed invariant set not containing proper subset which would be closed and invariant is called minimal.

Definition 1.8. A point $p \in X$ is called ω-limit point of the motion $\pi(\cdot, x)$ and of the point $x \in X$ if there exist a sequence $\{t_n\} \subset \mathbb{T}$ such that $t_n \to +\infty$ and $p = \lim_{n \to +\infty} \pi(t_n, x)$.

The set of all ω-limit points of the motion $\pi(\cdot, x)$ is denoted by ω_x and is called ω-limit set of this motion.

Definition 1.9. A point x and motion $\pi(\cdot, x)$ are called stable in the sense of Lagrange in positive direction and denoted st. L^+ if $H^+(x) := \overline{\Sigma}_x^+$ is a compact set, where $\Sigma_x^+ := \pi(\mathbb{T}_+, x)$ and $\mathbb{T}_+ := \{t \mid t \in \mathbb{T},\ t \geq 0\}$.

Definition 1.10. A point x and motion $\pi(\cdot, x)$ are called stable in the sense of Lagrange and denoted st. L if $H(x) := \overline{\Sigma}_x$ is a compact set, where $\Sigma_x := \pi(\mathbb{T}, x)$.

Definition 1.11. A point $x \in X$ is called fixed point or stationary point if $xt = x$ for all $t \in \mathbb{T}$ and τ-periodic if $x\tau = x$ ($\tau > 0,\ \tau \in \mathbb{T}$).

Definition 1.12. Let $\varepsilon > 0$. A number $\tau \in \mathbb{T}$ is called ε-shift (ε-almost period) of x if $\rho(x\tau, x) < \varepsilon$ ($\rho(x(t+\tau), xt) < \varepsilon$ for all $t \in \mathbb{T}$).

Definition 1.13. A point $x \in X$ is called almost recurrent (almost periodic) if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that on every segment from $\mathbb{T}$ of length l there exists ε-shift (ε-almost period) of the point x.

Definition 1.14. If a point $x \in X$ is almost recurrent and the set $H(x) = \overline{\Sigma}_x$ is compact, then the point x is called recurrent.

Definition 1.15. A point $x \in X$ is called positively Poisson stable if $x \in \omega_x$.

Definition 1.16. The motion $\pi(\cdot, x) : \mathbb{T} \to X$ of the semigroup dynamical system $(X, \mathbb{T}, \pi)$ is called continuable onto $\mathbb{S}$, if there exists a continuous mapping $\varphi : \mathbb{S} \to X$ such that $\pi^t \varphi(s) = \varphi(t+s)$ for all $t \in \mathbb{T}$ and $s \in \mathbb{S}$. In that case by α_φ we will denote the set $\{y \mid \exists t_n \to -\infty,\ t_n \in \mathbb{S}_-,\ \varphi(t_n) \to y\}$, where φ is an extension onto $\mathbb{S}$ of the motion $\pi(\cdot, x)$. The set α_φ is called α-limit set of φ and its points are called α-limit for φ.

Along with the dynamical system $(X, \mathbb{T}, \pi)$ let us consider $(Y, \mathbb{T}, \sigma)$, where Y is a complete metric space with metric d.

Definition 1.17. Following [100], one will say that the sequence $\{t_n\} \subset \mathbb{T}$ directs a point $x \in X$ to the point $p \in X$ if $p = \lim_{n\to+\infty} xt_n$. The sequence $\{t_n\}$ is called proper sequence of the point x if $x = \lim_{n\to+\infty} xt_n$.

By $\mathfrak{M}_{x,p}$ is denoted the set of all the sequences directing x to p, by $\mathfrak{N}_x$ -the set of all proper sequences of the point x (i.e., $\mathfrak{N}_x := \mathfrak{M}_{x,x}$) and $\mathfrak{M}_x := \cup\{\mathfrak{M}_{x,p} : p \in X\}$.

Definition 1.18. A point $x \in X$ is called [100] comparable by the character of recurrence with $y \in Y$ or, in short, comparable with y if for every $\varepsilon > 0$ there exists $\delta > 0$ such that δ-shift of the point y is ε-shift for $x \in X$.

In the work [100], it is shown that the point $x \in X$ is comparable with $y \in Y$ if and only if $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Definition 1.19. A point $x \in X$ is called [100] uniformly comparable by the character of recurrence with $y \in Y$ or, in short, uniformly comparable with y if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t \in \mathbb{T}$ every δ-shift of the point yt is ε-shift for xt, that is, $\delta > 0$ is such that for every two numbers $t_1, t_2 \in \mathbb{T}$ for which $d(yt_1, yt_2) < \delta$ is held the inequality $\rho(xt_1, xt_2) < \varepsilon$.

In the case when $y \in Y$ is stable in the sense of Lagrange (i.e., Σ_y is a relatively compact set) in [100] is proved that $x \in X$ is uniformly compared with $y \in Y$ if and only if $\mathfrak{M}_y \subseteq \mathfrak{M}_x$.

Definition 1.20. Points x_1 and x_2 from X are called positively proximal (distal) if

$$\inf\{\rho(x_1 t, x_2 t) : t \in \mathbb{T}_+\} = 0 \quad (\inf\{\rho(x_1 t, x_2 t) : t \in \mathbb{T}_+\} > 0). \tag{1.3}$$

Definition 1.21. A set $A \subseteq X$ is called [86] uniformly Lyapunov stable in positive direction with respect to the set $B \subseteq X$ (denotation-un. st. $\mathcal{L}^+B$) if $A \subseteq \overline{B}$ and for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that the inequality $\rho(p, r) < \delta$ $(p \in A,\ r \in B)$ implies $\rho(pt, rt) < \varepsilon$ for all $t \in \mathbb{T}_+$.

Definition 1.22. Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ $(\mathbb{S}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2 \subseteq \mathbb{S})$ be two dynamical systems. The mapping $h : X \to Y$ is called homomorphism (resp., isomorphism) of the dynamical system $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ if the mapping h is continuous (resp., homeomorphic) and $h(\pi(t, x)) = \sigma(t, h(x))$ for all $x \in X$ and $t \in \mathbb{T}_1$. In that case $(X, \mathbb{T}_1, \pi)$ is called an extension of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ and $(Y, \mathbb{T}_2, \sigma)$ is the factor of $(X, \mathbb{T}_1, \pi)$. The dynamical system $(Y, \mathbb{T}_2, \sigma)$ is also called (see, e.g., [93, 100]) base of the extension $(X, \mathbb{T}_1, \pi)$.

Definition 1.23. The triplet $((X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h)$, where h is a homomorphism of $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$, we will call nonautonomous dynamical system.

Let $t \in \mathbb{T}$. Denote mapping $\pi^t : X \to X$ by the equality $\pi^t(x) = \pi(t,x)$. If $\mathcal{F} \subseteq \mathbb{T}$ and $M \subseteq X$, then assume $E^*(M,\mathcal{F}) := \overline{\{\pi^t|_M : t \in \mathcal{F}\}}$ where by the line is denoted the closure in X^M and $E(M,\mathcal{F}) := \{\xi \mid \xi \in E^*(M,S),\ \xi(M) \subseteq M\}$.

Definition 1.24. A dynamical system $(X\times Y,\mathbb{T},\lambda)$ is called direct product of the dynamical systems $(X,\mathbb{T},\pi)$ and $(Y,\mathbb{T},\sigma)$ if $\lambda(t,(x,y)) = (\pi(t,x),\sigma(t,y))$ for all $(x,y) \in X \times Y$ and $t \in \mathbb{T}$.

1.2. Poisson Asymptotically Stable Motions

Definition 1.25. A motion $\pi(\cdot,x)$ is called asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent, asymptotically Poisson stable) if there exists a stationary (resp., τ-periodic, almost periodic, recurrent, Poisson stable) motion $\pi(\cdot,p)$ such that

$$\lim_{t\to+\infty} \rho(xt,pt) = 0. \tag{1.4}$$

Denote by $P_x := \{p \mid p \in \omega_x \cap \omega_p, \lim_{t\to+\infty} \rho(xt,pt) = 0\}$. It is clear that the motion $\pi(\cdot,x)$ is asymptotically Poisson stable if and only if $P_x \neq \emptyset$. From the definition of asymptotical Poisson stability, generally speaking, does not follow that P_x consists from a single point. The lemma below points out a simple condition with which P_x consists strictly from one point.

Lemma 1.26. *Let $x \in X$ be asymptotically Poisson stable. If points from ω_x are mutually distal in positive direction, then P_x consists from one point.*

Proof. In virtue of asymptotical Poisson stability of $\pi(\cdot,x)$ the set P_x is not empty. Assume that in P_x there are two different points p_1 and p_2. Under the conditions of Lemma 1.26 p_1 and p_2 are positively distal. On the other hand from (1.4) we have

$$\lim_{t\to+\infty} \rho(p_1 t, p_2 t) = 0. \tag{1.5}$$

The equality (1.5) contradicts to the positive distality of the points p_1 and p_2. The lemma is proved. □

Lemma 1.27. *Let $x \in X$ be almost periodic. Then the following statements hold:*

(1) *for every $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that on every segment of length l from $\mathbb{T}$ there is a number τ such that $\rho(p(t+\tau),pt) < \varepsilon$ for all $p \in H(x)$ and $t \in \mathbb{T}$;*
(2) *$H(x)$ is uniformly Lyapunov stable (in positive direction) with respect to $H(x)$.*

Proof. Let $x \in X$ be almost periodic, $p \in H(x)$ and $\varepsilon > 0$. Then there exists $l = l(\varepsilon/2) > 0$ such that on every segment of length l from $\mathbb{T}$ there is a number τ for which

$$\rho(x(t+\tau),xt) < \frac{\varepsilon}{2} \tag{1.6}$$

for all $t \in \mathbb{T}$. For $p \in H(x)$ there exists a sequence $\{t_n\} \subset \mathbb{T}$ such that $p = \lim_{n\to+\infty} xt_n$. From (1.6) it follows that

$$\rho(x(t+\tau+t_n), x(t+t_n)) < \frac{\varepsilon}{2} \tag{1.7}$$

for all $t \in \mathbb{T}$ and $n \in \mathbb{N}$. Passing to the limit in (1.7) as $n \to +\infty$, we obtain

$$\rho(p(t+\tau), pt) < \varepsilon \tag{1.8}$$

for all $t \in \mathbb{T}$ and $p \in H(x)$. The statement (1) is proved.

Let us prove the second statement of the lemma. Let $\varepsilon > 0$ and $l = l(\varepsilon/2)$ be the same that in the proof of statement (1) of Lemma 1.27. Since the set $H(x)$ is compact, then on $H(x)$ the integral continuity is uniform. This means that for $\varepsilon/3$ and $l(\varepsilon/3)$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\rho(pt, qt) < \frac{\varepsilon}{3} \tag{1.9}$$

for all $t \in [0, l]$ as soon as $\rho(p, q) < \delta$ $(p, q \in H(x))$. Let now $t \geq l$, p and $q \in H(x)$ and $\rho(p, q) < \delta$. Then on the segment $[t - l, t] \subset \mathbb{T}$ there is a number τ such that

$$\rho(r(t+\tau), rt) < \frac{\varepsilon}{3} \tag{1.10}$$

for all $r \in H(x)$ and $t \in \mathbb{T}$. Present the number t as $s + \tau$, where $s \in [0, l]$. Then for $\bar{t} = s + \tau$

$$\begin{aligned}\rho(p\bar{t}, q\bar{t}) &= \rho(p(s+\tau), q(s+\tau)) \\ &\leq \rho(p(s+\tau), ps) + \rho(ps, qs) + \rho(qs, q(s+\tau)).\end{aligned} \tag{1.11}$$

From the last inequality and inequalities (1.9) and (1.10) it follows that

$$\rho(pt, qt) < \varepsilon \tag{1.12}$$

for all $t \geq l$. From (1.9) and (1.12) we obtain the second statement of lemma. □

Lemma 1.28. *Let the point $x \in X$ be almost periodic, then on ω_x the dynamical system $(X, \mathbb{T}, \pi)$ is distal, that is,*

$$\inf\{\rho(pt, qt) : t \in \mathbb{T}\} > 0 \tag{1.13}$$

for all $p, q \in H(x)$ $(p \neq q)$.

Proof. Assume that the statement of Lemma 1.28 does not take place. Then there exist $p, q \in \omega_x = H(x)$ $(p \neq q)$ and $t_n \in \mathbb{T}$ such that

$$\rho(pt_n, qt_n) \longrightarrow 0 \tag{1.14}$$

as $n \to +\infty$. According to Lemma 1.27 $H(x)$ is un. st. $\mathcal{L}$ with respect to $H(x)$ and, consequently, for the number $0 < \varepsilon < \rho(p, q)$ there is $\delta = \delta(\varepsilon/3)$ such that

$$\rho(pt, qt) < \frac{\varepsilon}{3} \tag{1.15}$$

for all $t \in \mathbb{T}$ as soon as $\rho(p,q) < \delta$ $(p,q \in H(x))$. From (1.14) it follows that for n big enough $\rho(pt_n, qt_n) < \delta$ and, consequently,

$$\rho(p(t_n + t), q(t_n + t)) < \frac{\varepsilon}{3} \tag{1.16}$$

for all $t \in \mathbb{T}$. By the number $\varepsilon/3$ and $t_n \in \mathbb{T}$ we chose $\tau \geq t_n$ such that

$$\rho(r\tau, r) < \frac{\varepsilon}{3} \tag{1.17}$$

for all $r \in H(x)$ (according to Lemma 1.27 such τ there exists). Then,

$$\rho(p,q) \leq \rho(p\tau, p) + \rho(p\tau, q\tau) + \rho(q\tau, q) \tag{1.18}$$

and according to (1.16) and (1.17) $\rho(p,q) < \varepsilon$. This fact contradicts to the choice of ε. The lemma is proved. □

Corollary 1.29. *If the motion $\pi(\cdot, x)$ is asymptotically almost periodic, then P_x consists of a single point.*

Proof. The formulated statement follows from Lemmas 1.26 and 1.28. □

Remark 1.30. (1) In the case of asymptotical almost periodicity the point p, figuring in the definition of asymptotical almost periodicity is defined uniquely.

(2) If the a motion is asymptotically recurrent but not asymptotically almost periodic, then the statement formulated above (Corollary 1.29), generally speaking, does not take place.

1.3. Criterion of Asymptotical Almost Periodicity

Theorem 1.3.1. *The point $x \in X$ is asymptotical stationary (resp., asymptotically τ-periodic, asymptotically almost periodic) if and only if the following conditions hold:*

(1) *x is st. L^+;*
(2) *Σ_x^+ is un. st. $\mathcal{L}^+\Sigma_x^+$;*
(3) *ω_x coincides with the stationary point (resp., τ-periodic trajectory, closure of the almost periodic trajectory).*

Proof. Necessity. Let the point x be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic). Then there exists a stationary (resp., τ-periodic, almost periodic) point p such that equality (1.4) takes place. From equality (1.4) it follows that x is st. L^+ (since an almost period point is st. L^+) and $\omega_x = \omega_p = H(p)$. So, we only must show that Σ_x^+ is un. st. $\mathcal{L}^+\Sigma_x^+$. From equality (1.4) and almost periodicity of the point p it follows that for every $\varepsilon > 0$ there exist numbers $\beta \geq 0$ and $l > 0$ such that on every segment of length l there is a number τ for which

$$\rho(x(t + \tau), xt) < \varepsilon \tag{1.19}$$

for all $t \geq \beta$ and $t + \tau \geq \beta$.

Let $\varepsilon > 0$. From the said above it follows that for the number $\varepsilon/3$ there exists a pair of numbers $\beta(\varepsilon/3)$ and $l(\varepsilon/3)$ such that on every segment of length $l(\varepsilon/3)$ there is a number τ for which

$$\rho(x(t+\tau), xt) < \frac{\varepsilon}{3} \tag{1.20}$$

for all $t \geq \beta$ and $t + \tau \geq \beta$. On the compact set $H^+(x)$ the continuity integral is uniform. Therefore, there exists $\gamma(\varepsilon) > 0$ such that for every points $x_1, x_2 \in \Sigma_x^+$ from the inequality $\rho(x_1, x_2) < \gamma$ it follows that

$$\rho(x_1 t, x_2 t) < \frac{\varepsilon}{3} \tag{1.21}$$

for all $t \in [\beta, \beta + l]$. Note that γ can be chosen smaller than ε. Let x_1 and x_2 be from Σ_x^+, that is, $x_i = xt_i$ ($t_i \in \mathbb{T}_+$, $i = 1, 2$). Then,

$$\rho(x_1 t, x_2 t) \leq \rho(x_1(t+\tau), x_1 t) + \rho(x_1(t+\tau), x_2(t+\tau)) + \rho(x_2(t+\tau), x_2 t). \tag{1.22}$$

Choose $\tau \in [\beta - t, \beta - t + l] \subset \mathbb{T}_+$, then from the last inequality and inequalities (1.20) and (1.21) it follows that

$$\rho(x_1 t, x_2 t) < \gamma \tag{1.23}$$

for all $t \geq \beta$. In virtue of the uniform integral continuity on Σ_x^+ the numbers β and γ ($\gamma < \varepsilon$) we can choose $\delta < \gamma$ so that from the inequality $\rho(x_1, x_2) < \delta$ ($x_1, x_2 \in \Sigma_x^+$) would follow $\rho(x_1 t, x_2 t) < \gamma$ for all $t \in [0, \beta]$. Let now $\rho(x_1, x_2) < \delta$, ($x_1, x_2 \in \Sigma_x^+$, $\delta < \gamma < \varepsilon$) and $t \in \mathbb{T}_+$. Then $\rho(x_1 t, x_2 t) < \varepsilon$.

Sufficiency. Let x be st. L^+, Σ_x^+ be un. st. $\mathcal{L}^+\Sigma_x^+$, and let ω_x coincide with the stationary point (resp., τ-periodic trajectory, closure of the almost periodic trajectory). Under the conditions of Theorem 1.3.1 for every natural n there exist $\beta_n \geq 0$, $l_n > 0$, and $\tau_n \in [n, n + l_n]$ such that

$$\rho(x(t+\tau_n), xt) < \frac{1}{n} \tag{1.24}$$

for all $t \geq \beta_n$ and $t+\tau_n \geq \beta_n$. By the L^+ stability of x the sequence $\{x\tau_n\}$ can be considered convergent. Assume $p := \lim_{n\to+\infty} x\tau_n$, then $p \in \omega_x$ and by Lemma 1.27 the point p is almost periodic.

Let us show that the sequence $\{x\tau_n\}$ converges to p uniformly on $\mathbb{T}_+$, that is,

$$\lim_{n\to+\infty} \sup\{\rho(x(t+\tau_n), pt) : t \in \mathbb{T}_+\} = 0. \tag{1.25}$$

In fact, since $\{x\tau_n\}$ is convergent, it is fundamental. Let $\varepsilon > 0$ and $\delta(\varepsilon) > 0$ be chosen from the uniform stability $\mathcal{L}^+\Sigma_x^+$ of the set Σ_x^+. Then there exists $N(\varepsilon) > 0$ such that $\rho(x\tau_n, x\tau_m) < \delta$ for all $n, m \geq N(\varepsilon)$ and, consequently,

$$\rho(x(t+\tau_n), x(t+\tau_m)) < \varepsilon \tag{1.26}$$

for all $t \in \mathbb{T}_+$. Passing to the limit in (1.26) as $m \to +\infty$ (for fixed $n \in \mathbb{N}$ and $t \in \mathbb{T}_+$), we obtain

$$\rho(x(t+\tau_n), pt) \leq \varepsilon \tag{1.27}$$

for all $t \in \mathbb{T}_+$ and $n \geq N(\varepsilon)$. Now we will show that $p \in P_x$. In fact,

$$\rho(xt, pt) \leq \rho(xt, x(t+\tau_n)) + \rho(x(t+\tau_n), pt) \leq \frac{1}{n} + \varepsilon \tag{1.28}$$

for all $n \geq N(\varepsilon)$ and $t \geq \beta_n$ and, consequently, $p \in P_x$. The theorem is proved. □

Lemma 1.31 (see [86]). *If the set A is un. st. $\mathcal{L}^+B$, then $\overline{A}$ is un. st. $\mathcal{L}^+\overline{B}$.*

Lemma 1.32 (see [86]). *If Σ_x^+ is un. st. $\mathcal{L}^+\Sigma_x^+$ and $\omega_x \neq \emptyset$, then ω_x is a minimal set.*

Remark 1.33. The statements of [86, Lemmas 1.3.2 and 1.3.3] are proved for group systems in the case when $\mathbb{T} = \mathbb{R}$. However, it is not difficult to verify that the reasoning in [86] allows us to prove these statements also in the case when $\mathbb{T} = \mathbb{Z}, \mathbb{R}_+$ or $\mathbb{Z}_+$.

Corollary 1.34. *If the point x is st. L^+ and Σ_x^+ is un. st. $L^+\Sigma_x^+$, then ω_x is a nonempty compact minimal set consisting of almost periodic motions.*

Lemma 1.35. *If x is st. L^+ and $t_n \to t_0$ ($t_0 \in \mathbb{T}$), then*

$$\lim_{n\to+\infty} \sup\{\rho(x(t+t_n), x(t+t_0)) : t \in \mathbb{T}\} = 0. \tag{1.29}$$

Proof. Let x be st. L^+ and $t_n \to t_0$. Suppose that equality (1.29) does not take place. Then there exist a subsequence $\{\bar{t}_n\}$ and $\varepsilon_0 > 0$ such that

$$\rho(x(t_n + \bar{t}_n), x(t_0 + \bar{t}_n)) \geq \varepsilon. \tag{1.30}$$

Since the point x is st. L^+, the sequence $\{\pi(x, \bar{t}_n)\}$ can be considered convergent. Put $\bar{x} = \lim_{n\to+\infty} x\bar{t}_n$. Then

$$\varepsilon_0 \leq \rho(x(t_n + \bar{t}_n), x(t_0 + \bar{t}_n)) = \rho((x\bar{t}_n)t_n, (x\bar{t}_n)t_0). \tag{1.31}$$

Passing to the limit in inequality (1.31) as $n \to +\infty$, we get the inequality $\varepsilon_0 \leq 0$, which contradicts to the choice of ε_0. The lemma is proved. □

Lemma 1.36. *If $(X, \mathbb{T}, \pi)$ is the group dynamical system ($\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$), then under the conditions of Theorem 1.3.1 the set ω_x is an almost periodic minimal set of $(X, \mathbb{T}_+, \pi)$ (i.e., ω_x is a minimal set and every point $p \in \omega_x$ is almost periodic).*

Proof. In fact, using Lemmas 1.32, 1.35, and some results from the work [101], we can show that every point $p \in \omega_x$ will be almost periodic in $(X, \mathbb{T}, \pi)$ too. □

Remark 1.37. Let $(X, \mathbb{T}, \pi)$ be a semigroup dynamical system ($\mathbb{T} = \mathbb{R}_+$ or $\mathbb{Z}_+$) and X be an almost periodic minimal set. According to the results [93, 101] on the space X there exists a unique group dynamical system $(X, \mathbb{S}, \tilde{\pi})$ such that

(1) $\tilde{\pi}|_{\mathbb{S}_+\times X} = \pi$;

(2) $(X, \mathbb{S}, \tilde{\pi})$ is an almost periodic minimal set.

Theorem 1.3.2. *The following statements are equivalent:*

(1) *the point $x \in X$ is asymptotically almost periodic;*
(2) *for every $\varepsilon > 0$ there exist numbers $\beta \geq 0$ and $l > 0$ such that on every segment of length l there is a number τ for which inequality* (1.19) *holds for every $t \geq \beta$ and $t + \tau \geq \beta$;*
(3) *the point x is st. L^+ and Σ_x^+ is un. st. $\mathcal{L}^+\Sigma_x^+$;*
(4) *from any sequence $t_n \to +\infty$ we can extract a subsequence $\{t_{k_n}\}$ such that $\{\pi(t_{k_n}, x)\}$ converges uniformly with respect to $t \in \mathbb{T}_+$, that is, there exists $p \in X$ such that* (1.25) *is fulfilled.*

Proof. Note that the implication (1)⇒(2) follows from the definition of asymptotical almost periodicity (see the proof of Theorem 1.3.1).

Let us show that (3) follows from (2). From condition 2. it follows that the point x is st. L^+. Really, let $\varepsilon > 0$. For the number $\varepsilon/2$ there exist numbers $\beta \geq 0$ and $l > 0$ such that on every segment of length l there is a number τ for which

$$\rho(x(t+\tau), xt) < \frac{\varepsilon}{2} \tag{1.32}$$

for all $t \geq \beta$ and $t + \tau \geq \beta$. Let us show that $M = \pi([\beta, \beta + l], x)$ approximates $Q = \{\pi(t, x) : t \geq \beta\}$ with the exactness of $\varepsilon/2$. Really, if $t \geq \beta$, then exists $\tau \in [\beta - t, \beta - t + l]$ such that (1.29) holds and, consequently, $Q \subseteq S(M, \varepsilon/2)$. Since the set M is closed and compact, it possesses a finite $\varepsilon/2$-net, which in virtue of the inclusion $\overline{Q} \subseteq \overline{S}(M, \varepsilon/2)$ is a $\varepsilon/2$-net of the set Q. As the space X is complete, the set Q is compact. It remains to note that $\Sigma_x^+ = \pi([0, \beta], x) \cup Q$. At last, from the proof of necessity of Theorem 1.3.1 it follows condition (2) and stability L^+ of the point x we give the uniform stability $L^+\Sigma_x^+$ of the set Σ_x^+.

Let x be st. L^+, Σ_x^+ be un. st. $\mathcal{L}^+\Sigma_x^+$ and $t_n \to +\infty$. Since the point x is st. L^+, from $\{t_n\}$ we can extract a subsequence $\{t_{k_n}\}$ such that $xt_{k_n} \to p$. In virtue of the uniform stability $\mathcal{L}^+\Sigma_x^+$ of the set Σ_x^+, reasoning as well as in the proof of necessity of Theorem 1.3.1, we obtain equality (1.25).

Let us show that (4) implies (3). Suppose the contrary, that is, that there exist a number $\varepsilon_0 > 0$, sequences $\delta_n \downarrow 0$, $\{t_n^{(i)}\}$ $(i = 1, 2)$, and $\{\bar{t}_n\}$ such that

$$\rho(xt_n^{(1)}, xt_n^{(2)}) < \delta_n, \qquad \rho(x(t_n^{(1)} + \bar{t}_n), x(t_n^{(2)} + \bar{t}_n)) \geq \varepsilon_0. \tag{1.33}$$

It is obvious that from (4) it follows that x is st. L^+, hence the sequence $\{xt_n^{(i)}\}$ $(i = 1, 2)$ can be considered convergent. Assume $\bar{x}_i = \lim_{n\to+\infty} xt_n^{(i)}$ $(i = 1, 2)$. From inequality (1.33) it follows that $\bar{x}_1 = \bar{x}_2 = \bar{x}$. Let us show that the sequence $\{xt_n^{(i)}\}$ converges to $\bar{x}$ uniformly w.r.t. $t \in \mathbb{T}_+$. Two cases are possible:

(a) the sequence $\{t_n^{(i)}\}$ is bounded and without loss of generality it can be considered convergent. Then the needed statement follows from Lemma 1.35;

(b) the sequence $\{t_n^{(i)}\}$ is unbounded. By (4) the sequence $\{xt_n^{(i)}\}$ can be considered convergent uniformly with respect to $t \in \mathbb{T}_+$. So, we showed that $\bar{x} = \lim_{n\to+\infty} xt_n^{(i)}$ $(i = 1, 2)$ and the convergence in the last equality is uniform with respect to $t \in \mathbb{T}_+$. Then for the number $\varepsilon_0/2$ there is a natural number n_0 such that

$$\rho\big(x(t_n^{(1)}+t),x(t_n^{(2)}+t)\big) < \frac{\varepsilon_0}{2} \tag{1.34}$$

for all $t \in \mathbb{T}_+$ and $n \geq n_0$. In particular, for $t = t_n$,

$$\rho\big(x(t_n^{(1)}+t_n),x(t_n^{(2)}+t_n)\big) < \frac{\varepsilon_0}{2} \tag{1.35}$$

Inequality (1.35) contradicts to inequality (1.33). The obtained contradiction proves the required statement.

At last, let us show that from (3) it follows (1). Let the point $x \in X$ be st. L^+ and Σ_x^+ be un. st. $\mathcal{L}^+\Sigma_x^+$. Then according to Corollary 1.34 the set ω_x is a nonempty compact minimal set consisting of almost periodic motions. By Theorem 1.3.1 the point x is asymptotically almost periodic. The theorem is completely proved. □

1.4. Asymptotically Periodic Motions

Theorem 1.4.1. *For the asymptotic τ-periodicity of the point $x \in X$ it is necessary and sufficient that the sequence $\{\pi(k\tau,x)\}_{k=0}^{+\infty}$ would be convergent.*

Proof. Necessity. Let the point $x \in X$ be asymptotically τ-periodic, that is, there exists a τ-periodic point p such that equality (1.4) takes place. Then

$$\rho\big(x(k\tau),p(k\tau)\big) = \rho\big(x(k\tau),p\big). \tag{1.36}$$

Passing to the limit in (1.36) as $k \to +\infty$ and taking into consideration (1.4), we will obtain the required result.

Sufficiency. Let $\{\pi(k\tau,x)\}_{k=0}^{+\infty}$ be convergent. Assume $p = \lim_{k\to+\infty} \pi(k\tau,x)$. Note that

$$p\tau = \Big(\lim_{k\to+\infty} x(k\tau)\Big)\tau = \lim_{k\to+\infty} \big(x(k\tau)\big)\tau = \lim_{k\to+\infty} x(k+1)\tau = p. \tag{1.37}$$

So, the point p is τ-periodic. Let us show that x is st. L^+. In fact, let $\{t_n\} \subset \mathbb{T}_+$. Then $t_k = m_k\tau + \bar{t}_k$ (m_k, $\bar{t}_k \in \mathbb{T}_+$, $\tau > 0$ and $\bar{t}_k \in [0,\tau)$). The sequence $\{\bar{t}_k\}$ can be considered convergent and let $t_0 := \lim_{k\to+\infty} \bar{t}_k$. Then $\lim_{k\to+\infty} xt_k = \lim_{k\to+\infty} x(m_k\tau + \bar{t}_k) = \lim_{k\to+\infty}(x(m_k\tau))\bar{t}_k = pt_0$.

In virtue of stability L^+ of x the integral continuity on the set $H^+(x)$ is uniform. Taking in consideration also the fact that $\lim_{k\to+\infty} x(k\tau) = p$, we get

$$\lim_{k\to+\infty} \sup\{\rho(x(k\tau+t),pt) : t \in [0,\tau]\} = 0 \tag{1.38}$$

and, consequently,

$$\rho(xt,pt) = \rho\big(x(k\tau+\tilde{t}),p\tilde{t}\big) \leq \sup\{\rho\big(x(k\tau+\tilde{t}),p\tilde{t}\big) \mid \tilde{t} \in [0,\tau]\}, \tag{1.39}$$

where $t = k\tau + \tilde{t}$ ($k = [t]$, $\tilde{t} \in [0,\tau]$). Passing to the limit in (1.39) as $t \to +\infty$ ($k = [t] \to +\infty$ as $t \to +\infty$), we obtain equality (1.4). The theorem is proved. □

Corollary 1.38. *The point x is asymptotically stationary if and only if the sequence $\{x(k\tau)\}_{k=0}^{\infty}$ converges for every $\tau \in \mathbb{T}_+$.*

Theorem 1.4.2. *Let $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$. The point x is asymptotically stationary if and only if the sequence $\{\pi(t_n, x)\}$ converges, where $t_n := \sum_{k=1}^{n} 1/k$.*

Proof. The necessity is obvious, let us prove the sufficiency. Let the point $x \in X$ be such that the sequence $\{xt_n\}$, where $t_n = \sum_{k=1}^{n} 1/k$, is convergent. Put $p = \lim_{n\to+\infty} xt_n$ and show that

$$\lim_{t\to+\infty} \rho(xt, p) = 0. \tag{1.40}$$

It is obvious that to prove (1.40) it is enough to prove that for every sequence $\{t'_k\} \subset \mathbb{T}$, $t'_k \to +\infty$, there takes place

$$\lim_{k\to+\infty} \rho(xt'_k, p) = 0. \tag{1.41}$$

By the sequence $\{t'_k\}$ we will define the sequence

$$t_{n_k} := \max\{t_n \mid t_n \le t'_k\}. \tag{1.42}$$

The sequence $\{t_{n_k}\}$ defined by (1.42) possesses the next property:

$$t_{n_k} \le t'_k < t_{n_k+1}. \tag{1.43}$$

From (1.43), it follows that $0 \le t'_k - t_{n_k} < t_{n_k+1} - t_{n_k} = 1/(n_k+1)$. From the last inequality it follows that $\lim_{k\to+\infty}(t'_k - t_{n_k}) = 0$. It remains to note that $\pi(t'_k, x) = \pi(t'_k - t_{n_k}, \pi(t_{n_k}, x))$. Since $\{xt_{n_k}\}$ and $\{t'_k - t_{n_k}\}$ converges, then the sequence $\{xt'_k\}$ is convergent too, and obviously $\lim_{k\to+\infty} xt'_k = p$. The theorem is proved. □

Remark 1.39. If $(X, \mathbb{T}, \pi)$ is a cascade ($\mathbb{T} = \mathbb{Z}_+$ or $\mathbb{Z}$) and the point $p \in X$ is m-periodic, then obviously the set $\{p, \pi(1, p), \ldots, \pi((m-1), p)\}$ is its trajectory consisting from exactly m different points.

Lemma 1.40 (see [102]). *Let $\mathbb{T} = \mathbb{Z}_+$ or $\mathbb{Z}$ and $x \in X$ be a st. L^+ point. The set ω_x consists of a finite number of points if and only if there exists an m-periodic point $p \in X$ such that $\omega_x = \{p, \pi(1, p), \ldots, \pi(m-1, p)\}$.*

Theorem 1.4.3. *Let $\mathbb{T} = \mathbb{Z}_+$ or $\mathbb{Z}$. The point x is asymptotically m-periodic if and only if the point x is st. L^+ and its ω-limit set ω_x consists of exactly m different points.*

Proof. The necessity of the formulated statement directly follows from the definition of asymptotical m-periodicity and Remark 1.39.

Sufficiency. Let the point $x \in X$ be st. L^+ and $\omega_x = \{p_1, p_2, \ldots, p_m\}$, and $p_i \ne p_j$ as $i \ne j$, $i, j = 1, 2, \ldots, m$. Then

$$\lim_{n\to+\infty} \rho(xn, \omega_x) = 0. \tag{1.44}$$

From equality (1.44) it follows that

$$\delta(n) := \min \{\rho(xn, p_j n) : 1 \le j \le m\} \longrightarrow 0 \tag{1.45}$$

as $n \to +\infty$. Let $\rho_0 > 0$ be such that

$$\rho(p_i n, p_j n) \ge \rho_0 \tag{1.46}$$

for all $n \in \mathbb{T}$ and $i, j = 1, 2, \ldots, m$, $i \ne j$. Since the point x is st. L^+ on the set $H^+(x)$, there is held the condition of the uniform integral continuity. Let us choose for the number $\rho_0/3$ a number $\gamma(\rho_0/3) > 0$, $\gamma(\rho_0/3) < \rho_0/3$ from the condition of the uniform integral continuity. Then

$$\rho(\pi(\overline{x}, 1), \pi(p, 1)) < \frac{\rho_0}{3} \quad (\overline{x}, p \in H^+(x)), \tag{1.47}$$

as soon as $\rho(\overline{x}, p) < \gamma$. From (1.45), it follows that for $\gamma(\rho_0/5)$ there is n_0 such that $\delta(n) < \gamma(\rho_0/5)$ for all $n \ge n_0$. Then there exists $j_0 \in [1, m] \subset \mathbb{T}$ such that

$$\rho(xn_0, p_{j_0} n_0) < \gamma\left(\frac{\rho_0}{5}\right). \tag{1.48}$$

Assume

$$\Delta = \sup\left\{\widetilde{\Delta} \mid \rho(xn, p_{j_0} n) < \gamma\left(\frac{\rho_0}{5}\right),\ n \in \left[n_0, n_0 + \widetilde{\Delta}\right]\right\}. \tag{1.49}$$

(a) If $\Delta = +\infty$, then $\rho(xn, p_{j_0} n) < \gamma(\rho_0/5)$ for all $n \in \mathbb{N}$ and, consequently. for all $j \ne j_0$

$$\rho(xn, p_j n) \ge \rho(p_j n, p_{j_0} n) - \rho(xn, p_{j_0} n) \ge \rho_0 - \frac{\rho_0}{3} = \frac{2\rho_0}{3} > \gamma\left(\frac{\rho_0}{5}\right). \tag{1.50}$$

Therefore,

$$\rho(xn, p_{j_0} n) = \min_{1 \le j \le m} \rho(xn, p_j n) = \delta(n) \longrightarrow 0 \tag{1.51}$$

as $n \to +\infty$. And in this case the theorem is proved.

(b) Let us show that the case when $\Delta < +\infty$ is not possible. In fact, if we suppose that $\Delta < +\infty$ and put $n_0' = n_0 + \Delta$, then we have

$$\rho(xn_0', p_{j_0} n') < \gamma\left(\frac{\rho_0}{5}\right), \qquad \rho(x(n'+1), p_{j_0}(n'+1)) \ge \gamma\left(\frac{\rho_0}{5}\right), \tag{1.52}$$

$$\rho(x(n'+1), p_{j_0}(n'+1)) < \frac{\rho_0}{5}. \tag{1.53}$$

Since $\delta(n'+1) < \gamma(\rho_0/5)$, then there exists $p_{i_0} \ne p_{j_0}$ such that

$$\rho(x(n'+1), p_{i_0}(n'+1)) \ge \gamma\left(\frac{\rho_0}{5}\right) \tag{1.54}$$

and hence

$$\rho(p_{j_0}(n'+1), p_{i_0}(n'+1)) \\ \leq \rho(p_{j_0}(n'+1), x(n'+1)) + \rho(x(n'+1), p_{i_0}(n'+1)) < \gamma + \frac{\rho_0}{5} < \frac{\rho_0}{3} + \frac{\rho_0}{5} < \rho_0. \tag{1.55}$$

The last contradicts to the choice of the number ρ_0. The theorem is completely proved. □

Remark 1.41. The analogue of Theorem 1.4.2 for flows does not take place.

The said above is confirmed by the next example.

Example 1.42. Let us consider the dynamical system defined on the unit circle by the following rule. Let the center of the circle be a stationary point, the boundary of the circle be the trajectory of the periodic motion with the period $\tau = 1$. The rest of motions will be not special. And besides we assume that every semitrajectory Σ_x^+ is not un. st. $\mathcal{L}^+\Sigma_x^+$ for every inner point x of the circle that is different from the center. The described dynamical system is given by the system of differential equations, which in polar coordinates is looks as following:

$$\dot{r} = (r-1)^2 \\ \dot{\varphi} = r. \tag{1.56}$$

It is easy to see that ω-limit set of the point x is a trajectory of 1-periodic point, but the point x itself is not asymptotically 1-periodic, since Σ_x^+ is not un. st. $\mathcal{L}^+\Sigma_x^+$ (see Theorem 1.3.1).

1.5. Asymptotically Almost Periodic Functions

1.5.1. Dynamical Systems of Shifts in the Spaces of Continuous Functions

Below we give one general method of constructing of dynamical system in the spaces of continuous functions. The given method is used while getting many known dynamical systems in functional spaces (see, e.g., [92, 93, 100]).

Let $(X, \mathbb{T}, \pi)$ be a dynamical system on X, Y be a complete pseudometric space and $\mathcal{P}$ be its complete pseudometrics. By $C(X, Y)$ we denote the family of all continuous functions $f : X \mapsto Y$ endowed with the compact-open topology which is given by the following family of pseudometrics:

$$d_K^p(f, g) = \sup_{x \in K} p(f(x), g(x)) \quad (p \in \mathcal{P},\ K \in \mathcal{K}(X)), \tag{1.57}$$

where $\mathcal{K}(X)$ is the family of all compact subsets of X. Define for every $\tau \in \mathbb{T}$ a mapping $\sigma_\tau : C(X, Y) \mapsto C(X, Y)$ as follows: $(\sigma_\tau f)(x) = f(\pi(x, \tau))$ $(x \in X)$. Note the next properties of mapping σ_τ:

(a) $\sigma_0 = id_{C(X,Y)}$;

(b) $\sigma_{\tau_1} \circ \sigma_{\tau_1} = \sigma_{\tau_1+\tau_2}$;

(c) σ_τ is continuous.

As a rule, further we use the denotation $\sigma_\tau f = f^{(\tau)}$.

Lemma 1.43. *The mapping $\sigma : C(X,Y)\times\mathbb{T} \mapsto C(X,Y)$ defined by the equality $\sigma(f,\tau) = \sigma_\tau f$ ($f \in C(X,Y)$, $\tau \in \mathbb{T}$) is continuous.*

Proof. Let $f \in C(X,Y)$, $\tau \in \mathbb{T}$ and $\{f_\nu\}$, $\{\tau_\nu\}$ be arbitrary directedness converging to f and τ, respectively. Then for $K \in \mathcal{K}(X)$ we have

$$\begin{aligned} d_K^p&(\sigma(f_\nu,\tau_\nu),\sigma(f,\tau)) \\ &= \sup_{x\in K} p(\sigma(f_\nu,\tau_\nu)(x),\, \sigma(f,\tau)(x)) \\ &= \sup_{x\in K} p(f_\nu(\pi(x,\tau_\nu)),\, f(\pi(x,\tau))) \\ &\le \sup_{x\in K} p(f_\nu(\pi(x,\tau_\nu)),\, f(\pi(x,\tau_\nu))) + \sup_{x\in K} p(f(\pi(x,\tau_\nu)),\, f(\pi(x,\tau))) \\ &\le \sup_{x\in K, s\in Q} p(f_\nu(\pi(x,s)),\, f(\pi(x,s))) + \sup_{x\in K} p(f(\pi(x,\tau_\nu)),\, f(\pi(x,\tau))) \\ &\le \sup_{m\in\pi(K,Q)} p(f_\nu(m), f(m)) + \sup_{x\in K} p(f(\pi(x,\tau_\nu)),\, f(\pi(x,\tau))) \\ &= d^p_{\pi(K,Q)}(f_\nu, f) + \sup_{x\in K} p(f(\pi(x,\tau_\nu)),\, f(\pi(x,\tau))), \end{aligned} \tag{1.58}$$

where $Q = \overline{\{\tau_\nu\}}$. Passing to the limit in inequality (1.58) we get the necessary affirmation. □

Corollary 1.44. *$(C(X,Y),\mathbb{T},\sigma)$ is a dynamical system.*

Definition 1.45. The dynamical system $(C(X,Y),\mathbb{T},\sigma)$ is called a dynamical system of shifts (dynamical system of translations or dynamical system of Bebutov) in the space of continuous functions $C(X,Y)$.

Let us give some examples of dynamical systems of the type $(C(X,Y),\mathbb{T},\sigma)$ that are met in applications.

Example 1.46. Assume $X = \mathbb{T}$ and by $(X,\mathbb{T},\pi)$ denote a dynamical system on $\mathbb{T}$, where $\pi(x,t) = x+t$. The dynamical system $(C(\mathbb{T},Y),\mathbb{T},\sigma)$ is called a dynamical system of Bebutov [86, 92, 93, 99, 100]. The dynamical system of Bebutov is a useful means of study of general properties of continuous functions. Below, we use it to establish series of properties of almost periodic functions.

Example 1.47. Assume $X = \mathbb{T}\times W$, where W is some metric space and by $(X,\mathbb{T},\pi)$ we denote a dynamical system on $\mathbb{T}\times W$ defined by the following way $\pi((s,w),t) = (s+t,w)$.

The given above construction allows us in a natural way define on $C(\mathbb{T} \times W, Y)$ the dynamical system of shifts $(C(\mathbb{T} \times W, Y), \mathbb{T}, \sigma)$.

Example 1.48. Let $W = \mathbb{C}^n$, $Y = \mathbb{C}^m$, and $\mathcal{A}(\mathbb{T} \times \mathbb{C}^n, \mathbb{C}^m)$ be the set of all functions $f \in C(\mathbb{T} \times \mathbb{C}^n, \mathbb{C}^m)$ that are holomorphic with respect to the second argument. It is easy to check that the set $\mathcal{A}(\mathbb{T} \times \mathbb{C}^n, \mathbb{C}^m)$ is a closed invariant set of the dynamical system $(C(\mathbb{T} \times \mathbb{C}^n, \mathbb{C}^m), \mathbb{T}, \sigma)$ and, consequently, on $\mathcal{A}(\mathbb{T} \times \mathbb{C}^n, \mathbb{C}^m)$ there is induced a dynamical system $(\mathcal{A}(\mathbb{T} \times \mathbb{C}^n, \mathbb{C}^m), \mathbb{T}, \sigma)$.

1.5.2. Asymptotically Almost Periodic Functions of Fréchet.

In this section we will give the Fréchet definition [1, 2] of asymptotical almost periodicity of continuous functions and also some their properties. Let $\mathfrak{B}$ be a Banach space with the norm $|\cdot|$. In the space $C(\mathbb{R}, \mathfrak{B})$ with the help of the metric of Bebutov we can define the compact-open topology

$$\rho(\varphi, \psi) = \sup_{L>0} \min \Big\{ \max_{|t| \le L} |\varphi(t) - \psi(t)| ; L^{-1} \Big\}. \tag{1.59}$$

Consider the dynamical system of Bebutov $(C(\mathbb{R}, \mathfrak{B}), \mathbb{R}, \sigma)$.

Definition 1.49. One will say that the function $\varphi \in C(\mathbb{R}, \mathfrak{B})$ possesses the property A, if the motion $\sigma(\cdot, \varphi)$ generated by the function φ possesses this property in the dynamical system $(C(\mathbb{R}, \mathfrak{B}), \mathbb{R}, \sigma)$.

In the quality of the property A there can stand stability L^+, uniform stability $\mathcal{L}^+$, periodicity, almost periodicity, asymptotical almost periodicity and so on.

Note that the equality

$$\lim_{t \to +\infty} \rho(\sigma(\varphi, t), \sigma(p, t)) = 0 \tag{1.60}$$

is equivalent to the equality

$$\lim_{t \to +\infty} |\varphi(t) - p(t)| = 0, \tag{1.61}$$

where φ, $p \in C(\mathbb{R}, \mathfrak{B})$.

From the remarks above it follows that the function $\varphi \in C(\mathbb{R}, \mathfrak{B})$ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) if and only if there exist functions p and ω from $C(\mathbb{R}, \mathfrak{B})$ such that

(a) $\varphi(t) = p(t) + \omega(t)$ for all $t \in \mathbb{R}$;
(b) $\lim_{t \to +\infty} |\omega(t)| = 0$;
(c) p is stationary (resp., τ-periodic, almost periodic, recurrent).

Here p is called the main part of φ, and ω is called correction.

Remark 1.50. From Corollary 1.29 it follows that the functions p and ω from the conditions (a), (b), and (c) are defined uniquely, if φ is asymptotically almost periodic.

From the above said and Theorems 1.3.1, 1.3.2, 1.4.1, and 1.4.2 we get the following statements.

Theorem 1.5.1. *The function $\varphi \in C(\mathbb{R}, \mathfrak{B})$ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic) if and only if the function φ is st. L^+, the set Σ_φ^+ is un. st. $\mathcal{L}^+\Sigma_\varphi^+$ and ω_φ consists of a stationary function (resp., trajectory of τ-periodic function, closure of the trajectory of almost periodic function).*

Theorem 1.5.2. *Let $\varphi \in C(\mathbb{R}, \mathfrak{B})$. The following statements are equivalent:*

(1) *the function φ is asymptotically almost periodic;*
(2) *φ is st. L^+ and Σ_φ^+ is un. st. $\mathcal{L}^+\Sigma_\varphi^+$;*
(3) *for every $\varepsilon > 0$ there exist numbers $\beta \geq 0$ and $l > 0$ such that on every segment of length l there is a number τ for which the inequality $|\varphi(t+\tau) - \varphi(t)| < \varepsilon$ is held for all $t \geq \beta$ and $t + \tau \geq \beta$;*
(4) *from any sequence $\{t_n\}$, $t_n \to +\infty$ there can be extract a subsequence $\{t_{n_k}\}$ such that the sequence $\{\varphi^{(t_{n_k})}\}$, where $\varphi^{(t_{n_k})}(t) = \varphi(t + t_{n_k})$ for all $t \in \mathbb{R}$, converges uniformly with respect to $t \in \mathbb{R}_+$.*

Theorem 1.5.3. *The function $\varphi \in C(\mathbb{R}, \mathfrak{B})$ is asymptotically τ-periodic (resp., asymptotically stationary) if and only if the sequence $\{\varphi^{(t_n)}\}$ converges in $C(\mathbb{R}, \mathfrak{B})$, where $t_n := n\tau$ (resp., $t_n := \sum_{k=1}^{n} 1/k$).*

Lemma 1.51. *Let $\varphi \in C(\mathbb{R}, \mathfrak{B})$. The following statements are equivalent:*

(1) *the function φ is st. L^+;*
(2) *the function φ is relatively compact on $\mathbb{R}_+$ (i.e., $\varphi(\mathbb{R}_+)$ is a relatively compact set) and uniformly continuous on $\mathbb{R}_+$.*

Corollary 1.52. *Every asymptotically almost periodic function is relatively compact and uniformly continuous on $\mathbb{R}_+$.*

Definition 1.53. Let $\varphi \in C(\mathbb{R}, \mathfrak{B})$. They say that the function φ has a average value $M\{\varphi\}$ on $\mathbb{R}_+$, if there exists a limit of the expression $(1/L)\int_0^L \varphi(t)dt$ as $L \to +\infty$. So,

$$M\{\varphi\} := \lim_{L\to+\infty} \frac{1}{L}\int_0^L \varphi(t)dt. \tag{1.62}$$

Lemma 1.54. *Let $\omega \in C(\mathbb{R}, \mathfrak{B})$ and* $\lim_{t\to+\infty} |\omega(t)| = 0$. *Then* $M\{\omega\} = 0$.

Proof. Let $\varepsilon > 0$. Then there exists $A > 0$ such that $|\omega(t)| < \varepsilon$ for all $t \geq A$ and, consequently, for $L > A$

$$\begin{aligned} \left|\frac{1}{L}\int_0^L \omega(t)dt\right| &= \left|\frac{1}{L}\int_0^A \omega(t)dt + \frac{1}{L}\int_A^L \omega(t)dt\right| \\ &\leq \frac{1}{L}\int_0^A |\omega(t)|dt + \frac{\varepsilon}{L}|L-A|. \end{aligned} \tag{1.63}$$

Passing to the limit in inequality (1.63) as $L \to +\infty$, we obtain

$$\limsup_{L\to+\infty} \left| \frac{1}{L} \int_0^L \omega(t)dt \right| \leq \varepsilon. \tag{1.64}$$

Since $\varepsilon > 0$ is arbitrary, from inequality (1.64) it follows that on $\mathbb{R}_+$ there exists average value of the function ω and that it equals to zero. The lemma is proved. □

Corollary 1.55. *Let $\varphi \in C(\mathbb{R}, \mathfrak{B})$ be asymptotically almost periodic. Then φ has average value on $\mathbb{R}_+$ and $M\{\varphi\} = M\{p\}$, where p is the main part of φ.*

Theorem 1.5.4. *If every function $\varphi_k \in C(\mathbb{R}, \mathfrak{B}_k)$ $(k = 1, 2, \ldots, m)$ is asymptotically almost periodic, then the function $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m) \in C(\mathbb{R}, \mathfrak{B}_1) \times C(\mathbb{R}, \mathfrak{B}_2) \times \cdots \times C(\mathbb{R}, \mathfrak{B}_m)$ is asymptotically almost periodic too.*

Proof. The formulated statement follows directly from Theorem 1.5.1. In fact, put $\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2 \times \cdots \times \mathfrak{B}_m$ and define the norm $x \in \mathfrak{B}$ by the equality $\|x\| := \sum_{k=1}^m |x_k|_k$, where $|\cdot|_k$ is the norm on $\mathfrak{B}_k$ $(k = 1, 2, \ldots, m)$. Then $\varphi \in C(\mathbb{R}, \mathfrak{B})$. Let $\varphi_k \in C(\mathbb{R}, \mathfrak{B}_k)$ $(k = 1, 2, \ldots, m)$ be asymptotically almost periodic and $t_n \to +\infty$. Then there exists a subsequence t_{l_n} such that $\{\varphi_k^{(t_{l_n})}\}$ uniformly converges on $\mathbb{R}_+$ to some function $\widetilde{\varphi}_k$ $(k = 1, 2, \ldots, m)$ and hence $\varphi^{(t_{l_n})} = (\varphi_1^{(t_{l_n})}, \varphi_2^{(t_{l_n})}, \ldots, \varphi_m^{(t_{l_n})})$ uniformly converges to the function $\widetilde{\varphi} = (\widetilde{\varphi}_1, \widetilde{\varphi}_2, \ldots, \widetilde{\varphi}_m) \in C(\mathbb{R}, \mathfrak{B})$ on $\mathbb{R}_+$. □

Corollary 1.56. *Let $\varphi_k \in C(\mathbb{R}, \mathfrak{B})$ $(k = 1, 2, \ldots, m)$ be asymptotically almost periodic. Then $\varphi := \varphi_1 + \varphi_2 + \cdots + \varphi_m \in C(\mathbb{R}, \mathfrak{B})$ is asymptotically almost periodic too.*

Proof. By Theorem 1.5.4 the function $\widetilde{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_m) \in C(\mathbb{R}, \mathfrak{B}^m)$ is asymptotically almost periodic, that is, there exist functions $\widetilde{p} = (p_1, p_2, \ldots, p_m) \in C(\mathbb{R}, \mathfrak{B}^m)$ and $\widetilde{\omega} = (\omega_1, \omega_2, \ldots, \omega_m) \in C(\mathbb{R}, \mathfrak{B}^m)$ such that $(p_1, p_2, \ldots, p_m) \in C(\mathbb{R}, \mathfrak{B}^m)$ is almost periodic,

$$\lim_{t\to+\infty} \left(|\omega_1(t)| + |\omega_2(t)| + \cdots + |\omega_m(t)| \right) = 0 \tag{1.65}$$

and $\widetilde{\varphi} = \widetilde{p} + \widetilde{\omega}$. Then the function $\varphi = \varphi_1 + \varphi_2 + \cdots + \varphi_m$ can be presented in the form $\varphi = p + \omega$, where $\omega := \omega_1 + \omega_2 + \cdots + \omega_m$ and $p := p_1 + p_2 + \cdots + p_m$. The function p is almost periodic in virtue of the almost periodicity of functions $p_1, p_2, \ldots, p_m$ and $|\omega(t)| \to 0$ as $t \to +\infty$ and therefore φ is asymptotically almost periodic. □

Theorem 1.5.5. *Let $\{\varphi_k\} \subset C(\mathbb{R}, \mathfrak{B})$ be a sequence of asymptotically almost periodic functions and $\varphi_k \to \varphi$ uniformly on $\mathbb{R}_+$ as $k \to +\infty$, that is,* $\lim_{t\to+\infty} \sup\{|\varphi_k(t) - \varphi(t)| \mid t \in \mathbb{R}_+\} = 0$. *Then φ also is asymptotically almost periodic.*

Proof. Let $\varepsilon > 0$ and $k(\varepsilon) \in \mathbb{N}$ be such that

$$|\varphi_k(t) - \varphi(t)| < \frac{\varepsilon}{3} \tag{1.66}$$

for all $t \in \mathbb{R}_+$ and $k \geq k(\varepsilon)$. Since the function $\varphi_{k(\varepsilon)}$ is asymptotically almost periodic, then for the number $\varepsilon/3$ there are numbers $\beta(\varepsilon) \geq 0$ and $l(\varepsilon) > 0$ such that on every segment of length $l(\varepsilon)$ from $\mathbb{R}$ there exists a number τ such that

$$|\varphi_{k(\varepsilon)}(t+\tau) - \varphi_{k(\varepsilon)}(t)| < \frac{\varepsilon}{3} \tag{1.67}$$

for all $t \geq \beta(\varepsilon)$ and $t + \tau \geq \beta(\varepsilon)$. From inequalities (1.66) and (1.67) it follows that

$$\begin{aligned} &|\varphi(t+\tau) - \varphi(t)| \\ &\quad \leq |\varphi(t+\tau) - \varphi_{k(\varepsilon)}(t+\tau)| + |\varphi_{k(\varepsilon)}(t+\tau) - \varphi_{k(\varepsilon)}(t)| + |\varphi_{k(\varepsilon)}(t) - \varphi(t)| \\ &\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned} \tag{1.68}$$

for all $t \geq \beta(\varepsilon)$ and $t + \tau \geq \beta(\varepsilon)$. The theorem is proved. □

Corollary 1.57. *Let $\{\varphi_k\} \subset C(\mathbb{R}, \mathfrak{B})$ be a sequence of asymptotically almost periodic functions and the series $\sum_{k=1}^{+\infty} \varphi_k$ converges uniformly with respect to $t \in \mathbb{R}_+$ and $S \in C(\mathbb{R}, \mathfrak{B})$ is the sum of this series. Then S is an asymptotically almost periodic function.*

Let $\mathcal{AP}(\mathbb{R}_+, \mathfrak{B}) := \{\varphi \mid \varphi \in C(\mathbb{R}_+, \mathfrak{B}), \varphi$ be asymptotically almost periodic$\}$ and

$$\|\varphi\| = \sup\{|\varphi(t) : t \in \mathbb{R}_+\}. \tag{1.69}$$

Theorem 1.5.6. *$\mathcal{AP}(\mathbb{R}_+, \mathfrak{B})$ is a linear space and by equality (1.69) there is defined a complete norm on $\mathcal{AP}(\mathbb{R}_+, \mathfrak{B})$, that is, $(\mathcal{AP}(\mathbb{R}_+, \mathfrak{B}), \|\cdot\|)$ is a Banach space.*

Proof. The linearity of the space $\mathcal{AP}(\mathbb{R}_+, \mathfrak{B})$ follows from Corollary 1.56. From Corollary 1.52 it follows that the right-hand side of equality (1.69) is a finite number for every function $\varphi \in \mathcal{AP}(\mathbb{R}_+, \mathfrak{B})$. At last, let us show that norm (1.69) is complete. Let $\{\varphi_k\} \subset \mathcal{AP}(\mathbb{R}_+, \mathfrak{B})$ be a fundamental sequence. Then it is fundamental also in the space $C(\mathbb{R}_+, \mathfrak{B})$ (with respect to the metric of Bebutov) and, consequently, $\{\varphi_k\}$ is convergent in $C(\mathbb{R}_+, \mathfrak{B})$. So, there exists a function $\varphi \in C(\mathbb{R}_+, \mathfrak{B})$ such that $\varphi_k \to \varphi$ uniformly on compact subset from $\mathbb{R}_+$.

Let now $\varepsilon > 0$. Since $\{\varphi_k\}$ is fundamental with respect to norm (1.69), there exists a number $N(\varepsilon) > 0$ such that

$$|\varphi_m(t) - \varphi_n(t)| < \varepsilon \tag{1.70}$$

for all $t \in \mathbb{R}_+$ and $m, n \geq N(\varepsilon)$. Let us fix $t \in \mathbb{R}_+$, $n \geq N(\varepsilon)$ and pass to the limit in inequality (1.70) as $m \to +\infty$. Then we get

$$|\varphi(t) - \varphi_n(t)| \leq \varepsilon \tag{1.71}$$

for all $t \in \mathbb{R}_+$ and $n \geq N(\varepsilon)$. So, $\varphi_n \to \varphi$ uniformly on $\mathbb{R}_+$ and by Theorem 1.5.5 the function φ is asymptotically almost periodic. The theorem is proved. □

Denote by $\mathcal{P}(\mathbb{R}, \mathfrak{B})$ the Banach space of all almost periodic functions from $C(\mathbb{R}, \mathfrak{B})$ with the norm

$$\|\varphi\| = \sup\{|\varphi(t) : t \in \mathbb{R}\}, \tag{1.72}$$

and by $C_0(\mathbb{R}, \mathfrak{B})$ the Banach space of all functions $\varphi \in C(\mathbb{R}_+, \mathfrak{B})$ satisfying the condition $\lim_{t\to+\infty} |\varphi(t)| = 0$ and endowed with norm (1.69).

Theorem 1.5.7. *The continuously differentiable asymptotically almost periodic function $\varphi \in \mathcal{AP}(\mathbb{R}, \mathfrak{B})$ has an asymptotically almost periodic derivative φ' if and only if it is uniformly continuous on $\mathbb{R}_+$.*

Proof. The necessity follows from Corollary 1.52. Sufficiency. Let $\varphi \in \mathcal{AP}(\mathbb{R}, \mathfrak{B})$ be continuously differentiable and φ' be uniformly continuous on $\mathbb{R}_+$. Consider the sequence $\{\varphi_n\} \subset \mathcal{AP}(\mathbb{R}, \mathfrak{B})$ defined by the equality

$$\varphi_n(t) = n\left[\varphi\left(t + \frac{1}{n}\right) - \varphi(t)\right] = n\int_0^{1/n} \varphi'(t+\tau)d\tau. \tag{1.73}$$

Note that

$$\begin{aligned} |\varphi_n(t) - \varphi'(t)| &= \left| n\int_0^{1/n} [\varphi'(t+\tau) - \varphi'(t)]d\tau \right| \\ &\leq \max_{0\leq\tau\leq 1/n} |\varphi'(t+\tau) - \varphi'(t)|, \end{aligned} \tag{1.74}$$

and hence $\varphi_n \to \varphi'$ uniformly on $\mathbb{R}_+$, since φ' is uniformly continuous on $\mathbb{R}_+$. According to Theorem 1.5.5 $\varphi' \in \mathcal{AP}(\mathbb{R}, \mathfrak{B})$. □

Lemma 1.58. *Let $\varphi \in C(\mathbb{R}, \mathfrak{B})$ be continuously differentiable, having a uniformly continuous derivative φ' and* $\lim_{t\to+\infty} |\varphi(t)| = 0$. *Then* $\lim_{t\to+\infty} |\varphi'(t)| = 0$.

Proof. Let $\{\varphi_n\} \subset C(\mathbb{R}, \mathfrak{B})$ be the sequence defined by equality (1.73). From inequality (1.74) it follows that $\varphi_n \to \varphi'$ uniformly on $\mathbb{R}_+$. Besides, from equality (1.73) it follows that

$$\lim_{t\to+\infty} |\varphi_n(t)| = 0 \tag{1.75}$$

for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then there exists a number $n(\varepsilon) \in \mathbb{N}$ such that

$$|\varphi'(t) - \varphi_n(t)| < \frac{\varepsilon}{2} \tag{1.76}$$

for all $t \in \mathbb{R}_+$ and $n \geq n(\varepsilon)$. From (1.75) it follows that for $n(\varepsilon)$ there is $L(\varepsilon) > 0$ such that

$$|\varphi_{n(\varepsilon)}(t)| < \frac{\varepsilon}{2} \tag{1.77}$$

for all $t \geq L(\varepsilon)$. Then

$$|\varphi'(t)| \leq |\varphi'(t) - \varphi_{n(\varepsilon)}(t)| + |\varphi_{n(\varepsilon)}(t)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{1.78}$$

for all $t \geq L(\varepsilon)$. The lemma is proved. □

Lemma 1.59. *Let $\varphi \in C(\mathbb{R}, \mathfrak{B})$ be asymptotically almost periodic (i.e., $\varphi = p + \omega$, where $p \in \mathcal{P}(\mathbb{R}, \mathfrak{B})$ and $\omega \in C_0(\mathbb{R}, \mathfrak{B})$) and uniformly continuous on $\mathbb{R}_+$. Then p and ω are continuously differentiable, $p' \in \mathcal{P}(\mathbb{R}, \mathfrak{B})$, $\omega' \in C_0(\mathbb{R}, \mathfrak{B})$ and $\varphi' = p' + \omega'$.*

Proof. Under the conditions of Lemma 1.59 along with the function φ, according to Theorem 1.5.7, its derivative φ' also is asymptotically almost periodic. Let $\{\tau_n\}$ be such that $\tau_n \to +\infty$ and $\varphi^{(\tau_n)} \to p$. Since φ is asymptotically almost periodic, the sequence $\{\varphi'(t + \tau_n)\}$ can be considered convergent. Assume $\widetilde{p}(t) = \lim_{n\to+\infty} \varphi'(t + \tau_n)$ and note that

$$\varphi(t + \tau_n) = \varphi(\tau_n) + \int_0^t \varphi'(\tau + \tau_n)d\tau. \tag{1.79}$$

Passing to the limit in equality (1.79) as $n \to +\infty$, we obtain

$$p(t) = p(0) + \int_0^t \widetilde{p}(\tau)d\tau. \tag{1.80}$$

Equality (1.80) implies that p is continuously differentiable and $p' = \widetilde{p}$. Since $\widetilde{p} \in \omega_{\varphi'}$ and φ' is asymptotically almost periodic, then $p' \in \mathcal{P}(\mathbb{R}, \mathfrak{B})$. Therefore $\omega = \varphi - p$ is uniformly continuous on $\mathbb{R}_+$ together with its derivative $\omega' = \varphi' - p'$ and by Lemma 1.58 $\omega' \in C_0(\mathbb{R}, \mathfrak{B})$. The lemma is proved. □

Theorem 1.5.8. *Let $\varphi \in \mathcal{AP}(\mathbb{R}, \mathfrak{B})$ (i.e., $\varphi = p + \omega$, where $p \in \mathcal{P}(\mathbb{R}, \mathfrak{B})$ and $\omega \in C_0(\mathbb{R}, \mathfrak{B})$) and $F(t) := \int_0^t \varphi(\tau)d\tau$ has a compact values on $\mathbb{R}_+$ (i.e., $F(\mathbb{R}_+)$ is a relatively compact set). The function F is asymptotically almost periodic if and only if the integral $\int_0^{+\infty} \omega(\tau)d\tau$ converges, that is, there exists a finite limit*

$$\lim_{t\to+\infty} \int_0^t \omega(\tau)d\tau. \tag{1.81}$$

Proof. Let $\varphi, F \in \mathcal{AP}(\mathbb{R}, \mathfrak{B})$, where $F(t) = \int_0^t \varphi(\tau)d\tau$, and $F = P + \Omega$ ($P \in \mathcal{P}(\mathbb{R}, \mathfrak{B})$ and $\Omega \in C_0(\mathbb{R}, \mathfrak{B})$). Since $F' = \varphi$, then according to Lemma 1.59 $P' = p$, $\Omega' = \omega$ and, consequently, $\Omega(t) = \Omega(0) + \int_0^t \omega(\tau)d\tau$. Since $|\Omega(t)| \to 0$ as $t \to +\infty$, then $\lim_{t\to+\infty} \int_0^t \omega(\tau)d\tau = -\Omega(0)$.

Conversely. Suppose that there exists a finite limit $\lim_{t\to+\infty} \int_0^t \omega(\tau)d\tau = c$. Let us choose the sequence $\{\tau_n\} \to +\infty$ such that $\varphi^{(\tau_n)} \to p$ and consider the sequence F_n defined as follows:

$$F_n(t) = \int_0^t \varphi(t + \tau)d\tau + F(\tau_n). \tag{1.82}$$

Since the function F has a compact values on $\mathbb{R}_+$, then the sequence $\{F(\tau_n)\}$ can be considered convergent. Put

$$A = \lim_{n\to+\infty} F(\tau_n). \tag{1.83}$$

Passing to the limit in equality (1.82) as $n \to +\infty$, we get $\widetilde{F}(t) = \int_0^t p(\tau)d\tau + A$. Besides, note that $F_n(t) \in \overline{F(\mathbb{R}_+)}$ for all $t \geq -\tau_n$, that is, the function F_n has a compact values on $[-\tau_n, +\infty[$ and, consequently, $\widetilde{F}$ has a compact values on $\mathbb{R}$ ($\widetilde{F}(t) \in \overline{F(\mathbb{R}_+)}$ for all $t \in \mathbb{R}$). Therefore $P(t) := \int_0^t p(\tau)d\tau = \widetilde{F}(t) - A$ has a compact values and hence it is almost periodic [92, 93, 100]. The function $F(t) = \int_0^t \varphi(\tau)d\tau$ can be presented in the form

$$F(t) = P(t) + c + \left[\int_0^t \omega(\tau)d\tau - c\right]. \tag{1.84}$$

Since $\lim_{t\to+\infty}[\int_0^t \omega(\tau)d\tau - c] = 0$ and $P + c$ is almost periodic, then F is asymptotically almost periodic. □

Lemma 1.60. *Let $\varphi_k \in \mathcal{A}P(\mathbb{R}, \mathfrak{B}_k)$ $(k = 1, 2, \ldots, m)$ and $\Phi \in C(Q, \mathfrak{B})$, where $Q = \overline{\varphi_1(\mathbb{R}_+)} \times \overline{\varphi_2(\mathbb{R}_+)} \times \cdots \times \overline{\varphi_m(\mathbb{R}_+)}$. The function φ defined by the equality*

$$\varphi(t) := \Phi(\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t)) \quad (t \in \mathbb{R}_+) \tag{1.85}$$

is asymptotically almost periodic.

Proof. In virtue of the asymptotical almost periodicity of the functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ the set $Q_+ = \overline{\varphi_1(\mathbb{R}_+)} \times \overline{\varphi_2(\mathbb{R}_+)} \times \cdots \times \overline{\varphi_m(\mathbb{R}_+)}$ is compact and, consequently, the function $\Phi \in C(Q, \mathfrak{B})$ is uniformly continuous on Q_+. Let p_k be an almost periodic function such that $\varphi_k = p_k + \omega_k$ for $\omega_k \in C_0(\mathbb{R}, \mathfrak{B}_k)$ $(k = 1, 2, \ldots, m)$. Then $\overline{p_k(\mathbb{R})} \subset \overline{\varphi_k(\mathbb{R}_+)}$ and hence $\widetilde{Q} = \overline{p_1(\mathbb{R})} \times \overline{p_2(\mathbb{R})} \times \cdots \times \overline{p_m(\mathbb{R})} \subset Q_+$ is a compact set.

Let $\varepsilon > 0$, $\delta(\varepsilon)$ be chosen from the uniform continuity of Φ on Q_+ and $L(\varepsilon) > 0$ be such that

$$|\varphi_k(t) - p_k(t)| < \delta(\varepsilon) \tag{1.86}$$

for all $t \geq L(\varepsilon)$ and $k = 1, 2, \ldots, m$. Assume $p(t) := \Phi(p_1(t), p_2(t), \ldots, p_m(t))$ and $\omega(t) := \varphi(t) - p(t)$. Note that the function $p \in C(\mathbb{R}, \mathfrak{B})$ is almost periodic, as the functions $p_k \in C(\mathbb{R}, \mathfrak{B}_k)$ $(k = 1, 2, \ldots, m)$ are almost periodic and Φ is uniformly continuous on $Q_+ \supset \widetilde{Q} = \overline{p_1(\mathbb{R})} \times \overline{p_2(\mathbb{R})} \times \cdots \times \overline{p_m(\mathbb{R})}$. Besides,

$$|\omega(t)| = |\Phi(\varphi_1(t), \ldots, \varphi_m(t)) - \Phi(p_1(t), \ldots, p_m(t))| < \varepsilon \tag{1.87}$$

for all $t \geq L(\varepsilon)$. The lemma is proved. □

Lemma 1.61. *Let $\{\varphi_k\} \subset \mathcal{A}P(\mathbb{R}, \mathfrak{B})$ and $\varphi = \lim_{k\to+\infty} \varphi_k$ in $\mathcal{A}P(\mathbb{R}, \mathfrak{B})$. Then $M\{\varphi\} = \lim_{k\to+\infty} M(\varphi_k)$.*

Proof. Let $\varepsilon > 0$ and $k(\varepsilon) > 0$ be such that

$$|\varphi_k(t) - \varphi(t)| < \varepsilon \tag{1.88}$$

for all $t \in \mathbb{R}_+$ and $k \geq k(\varepsilon)$. Since

$$|M\{\varphi_k\} - M\{\varphi\}| \leq M\{|\varphi_k(t) - \varphi(t)|\}, \tag{1.89}$$

then from (1.88) it follows the inequality

$$|M\{\varphi_k\} - M\{\varphi\}| \leq \varepsilon \tag{1.90}$$

holds for all $k \geq k(\varepsilon)$. The lemma is proved. □

Lemma 1.62. *If* $\varphi \in \mathcal{A}P(\mathbb{R}_+, \mathfrak{B})$, *then*

$$M\{\varphi\} = \lim_{T\to+\infty} \frac{1}{T} \int_t^{t+T} \varphi(s)ds. \tag{1.91}$$

And the limit (1.91) *exists uniformly on* $t \in \mathbb{R}_+$.

Proof. Let $\varphi \in \mathcal{A}P(\mathbb{R}_+, \mathfrak{B})$, $p \in \mathcal{P}(\mathbb{R}, \mathfrak{B})$, and $\omega \in C_0(\mathbb{R}_+, \mathfrak{B})$ such that $\varphi = p+\omega$. Then the equality

$$M\{p\} = \lim_{T\to+\infty} \frac{1}{T} \int_t^{t+T} p(s)ds \tag{1.92}$$

takes place uniformly with respect to $t \in \mathbb{R}$. Let $\varepsilon > 0$. Then for ω there is $L(\varepsilon) > 0$ such that

$$|\omega(t)| < \frac{\varepsilon}{2} \tag{1.93}$$

for all $t \geq L(\varepsilon)$ and, consequently,

$$\begin{aligned} &\left|\frac{1}{T}\int_0^T \omega(s+t)ds\right| \\ &\quad\leq \frac{1}{T}\int_0^{L(\varepsilon)} |\omega(s+t)|ds + \frac{1}{T}\int_{L(\varepsilon)}^T |\omega(s+t)|ds \leq \frac{\|\omega\|}{T}L(\varepsilon) + \frac{\varepsilon}{2}\frac{T-L(\varepsilon)}{T} \end{aligned} \tag{1.94}$$

for all $t \geq 0$, where $\|\omega\| = \sup\{|\varphi(t)| : t \in \mathbb{R}_+\}$. From (1.94) it follows that

$$\left|\frac{1}{T}\int_0^T \omega(s+t)ds\right| < \varepsilon \tag{1.95}$$

for all $t \in \mathbb{R}_+$ and $T > 2L(\varepsilon)\|\omega\|/\varepsilon$. Let $\varepsilon > 0$. Then for ω from (1.92) and (1.95) it follows (1.91). The lemma is proved. □

1.6. Asymptotically S^p Almost Periodic Functions

1.6.1. Dynamical Systems of Shifts in the Space $L^p_{\mathrm{loc}}(\mathbb{R}; \mathbf{B}; \mu)$.

Let $S \subseteq \mathbb{R}$, $(S, \mathbf{B}; \mu)$ be a space with measure and μ is the Radon measure, $\mathfrak{B}$-is a Banach space with the norm $|\cdot|$.

Definition 1.63. A function $f : S \to \mathfrak{B}$ is called [103] a step-function if it takes no more than a finite number of values. In this case, it is called measurable, if $f^{-1}(\{x\}) \in \mathbf{B}$ for every $x \in \mathfrak{B}$, and integrable if in addition $\mu(f^{-1}(\{x\})) < +\infty$. Then there is defined

$$\int f d\mu = \sum_{x \in \mathfrak{B}} \mu(f^{-1}(\{x\}))x. \tag{1.96}$$

The sum in the right-hand side of equality (1.96) is finite by assumption.

Definition 1.64. A function $f : S \to \mathfrak{B}$ is said to be measurable if there exists a sequence $\{f_n\}$ of step-functions measurable and such that $f_n(s) \to f(s)$ with respect to the measure μ almost everywhere.

Definition 1.65. A function $f : S \to \mathfrak{B}$ is called integrable, if there exists a sequence $\{f_n\}$ of step-functions, integrable and such that for every n the function $\varphi_n(s) = |f_n(s) - f(s)|$ is integrable and

$$\lim_{n \to +\infty} \int |f_n(s) - f(s)| d\mu(s) = 0. \tag{1.97}$$

Then $\int f_n d\mu$ converges in the space $\mathfrak{B}$ and its limit does not depend on the choice of the approximating sequence $\{f_n\}$ with the above mentioned properties. This limit is denoted by $\int f d\mu$ or $\int f(s) d\mu(s)$.

Let $1 \le p \le +\infty$. By $L^p(S; \mathfrak{B}, \mu)$ there is denoted the space of all measurable functions (classes of functions) $f : S \to \mathfrak{B}$ such that $|f| \in L^p(S; \mathbb{R}; \mu)$, where $|f|(s) = |f(s)|$. The space $L^p(S; \mathfrak{B}; \mu)$ is endowed with the norm

$$\|f\|_{L^p} = \left| \int \middle| f(s)|^p d\mu(s)\right|^{1/p}, \qquad \|f\|_\infty = \operatorname{supess} |f(s)|. \tag{1.98}$$

$L^p(S; \mathfrak{B}; \mu)$ with norm (1.98) is a Banach space.

Denote by $L^p_{\text{loc}}(\mathbb{R}; \mathfrak{B}; \mu)$ the set of all function $f : \mathbb{R} \to \mathfrak{B}$ such that $f_l \in L^p([-l, l]; \mathfrak{B}; \mu)$ for every $l > 0$, where f_l is the restriction of the function f onto $[-l, l]$.

Definition 1.66. The function $f : \mathbb{R} \to \mathfrak{B}$ is called decomposable, if for arbitrary $s \in \mathbb{R}$ one has $f(s) = \sum_{i=1}^N \varphi_i(s) g_i$, where $g_i \in \mathfrak{B}$ and φ_i is a scalar continuous function with the compact support $(i = 1, 2, \dots, N)$.

Lemma 1.67 (see [103]). *The following statements hold:*

(1) *every continuous functions $f : S \to \mathfrak{B}$ with compact support is integrable;*
(2) *in the space $L^p(\mathbb{R}; \mu; \mathfrak{B})$ the set of step-functions with compact support are dense.*

In the space $L^p_{\text{loc}}(\mathbb{R}; \mathfrak{B}; \mu)$ we define a family of seminorms $\|\cdot\|_{l,p}$ by the following rule:

$$\|f\|_{l,p} = \|f_l\|_{L^p([-l,l]; \mathfrak{B}; \mu)} \quad (l > 0). \tag{1.99}$$

Family of seminorms (1.99) defines a metrizable topology on $L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu)$. The metric that gives this topology can be defined, for instance, by the next equality

$$d_p(\varphi,\psi)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|\varphi-\psi\|_{n,p}}{1+\|\varphi-\psi\|_{n,p}}. \tag{1.100}$$

Let us define a mapping $\sigma : L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu)\times\mathbb{R}\to L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu)$ as follows: $\sigma(f,\tau)=f^{(\tau)}$ for all $f\in L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu)$ and $\tau\in\mathbb{R}$, where $f^{(\tau)}(s):=f(s+\tau)$ $(s\in\mathbb{R})$.

Lemma 1.68. $(L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu),\mathbb{R},\sigma)$ *is a dynamical system.*

Proof. It is enough to show the mapping σ is continuous. Let $f_n\to f$ in the space $L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu)$ and let $t_n\to t_0$. We will show that $\sigma(f_n,t_n)\to\sigma(f,t_0)$ as $n\to+\infty$, that is,

$$\left[\int_{|t|\le l}|f_n(t_n+s)-f(t_0+s)|^p d\mu(s)\right]^{1/p}\longrightarrow 0, \tag{1.101}$$

as $n\to+\infty$ for every $l>0$.

Note that

$$\begin{aligned}\left[\int_{|t|\le l}|f_n(t_n+s)-f(t_0+s)|^p d\mu(s)\right]^{1/p}\\ \le\left[\int_{|t|\le l}|f_n(t_n+s)-f(t_n+s)|^p d\mu(s)\right]^{1/p}+\left[\int_{|t|\le l}|f_n(t_n+s)-f(t_0+s)|^p d\mu(s)\right]^{1/p}.\end{aligned} \tag{1.102}$$

Besides, since $t_n\to t_0$, there exists $l_0>0$ such that $|t_n|\le l_0$ and $|t_n+s|\le|t_n|+|s|\le l_0+l=L$ for all $n=1,2,3,\ldots$, and, consequently,

$$\left[\int_{|s|\le l}|f_n(t_n+s)-f(t_n+s)|^p d\mu(s)\right]^{1/p}\le\left[\int_{|t|\le L}|f_n(t)-f(t)|^p d\mu(t)\right]^{1/p}\longrightarrow 0 \tag{1.103}$$

as $n\to+\infty$, since $f_n\to f$ in $L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu)$.

To estimate the second integral in the right-hand side of inequality (1.102) we will use Lemma 1.67. Let $\varepsilon>0$ and $g:\mathbb{R}\to\mathfrak{B}$ be a continuous function with the compact support such that

$$\left[\int_{|s|\le l+l_0}|g(s)-f(s)|^p d\mu(s)\right]^{1/p}\le\varepsilon. \tag{1.104}$$

Then,

$$\begin{aligned}
&\left[\int_{|t|\le l} |f(t+h_n)-f(t)|^p dt\right]^{1/p} \\
&\quad\le \left[\int_{|t|\le l} |f(t+h_n)-g(t+h_n)|^p dt\right]^{1/p} \\
&\qquad + \left[\int_{|t|\le l} |g(t+h_n)-g(t)|^p dt\right]^{1/p} + \left[\int_{|t|\le l} |f(t)-g(t)|^p dt\right]^{1/p} \\
&\quad\le 2\left[\int_{|t|\le L} |f(s)-g(s)|^p ds\right]^{1/p} + \left[\int_{|t|\le l} |g(t+h_n)-g(t)|^p dt\right]^{1/p} \\
&\quad\le 2\varepsilon + \max_{|t|\le l} |g(t+h_n)-g(t)| \cdot 2l
\end{aligned} \tag{1.105}$$

(where $L := l + l_0$ and $l_0 := \sup\{h_n \mid n \in \mathbb{N}\}$) and, consequently,

$$\lim_{n\to+\infty} \left[\int_{|t|\le l} |f(t+h_n)-f(t)|^p dt\right]^{1/p} \le 2\varepsilon \tag{1.106}$$

(because $\max_{|t|\le l} |g(t+h_n)-g(t)| \to 0$ as $h_n \to 0$). Since ε is arbitrary, from the last relation we obtain

$$\lim_{n\to+\infty} \left[\int_{|t|\le l} |f(t+h_n)-f(t)|^p dt\right]^{1/p} = 0. \tag{1.107}$$

From (1.102)–(1.107), it follows the continuity of the mapping σ. The lemma is proved. □

1.6.2. Stepanoff Asymptotically Almost Periodic Functions

Definition 1.69. A function $\varphi \in L^p_{\text{loc}}(\mathbb{R}; \mathfrak{B}; \mu)$ is called S^p almost periodic (almost periodic in the sense of Stepanoff [104]), if the motion $\sigma(\cdot, \varphi)$ is almost periodic in the dynamical system $(L^p_{\text{loc}}(\mathbb{R}; \mathfrak{B}; \mu), \mathbb{R}, \sigma)$. Analogously there is defined asymptotical S^p almost periodicity of functions.

Theorem 1.6.1. *Let $\varphi \in L^p_{\text{loc}}(\mathbb{R}; \mathfrak{B}; \mu)$. The following statements are equivalent:*

(1) *φ is S^p almost periodic;*
(2) *for every $\varepsilon > 0$ there exists $l > 0$ such that on every segment of length l in $\mathbb{R}$ there is a number τ for which*

$$\int_t^{t+1} |\varphi(s+\tau)-\varphi(s)|^p ds < \varepsilon^p \tag{1.108}$$

for all $t \in \mathbb{R}$;
(3) *φ is st. L and Σ_φ is un. st. $\mathcal{L}\Sigma_\varphi$ in the dynamical system $(L^p_{\text{loc}}(\mathbb{R}; \mathfrak{B}; \mu), \mathbb{R}, \sigma)$;*

(4) *from an arbitrary sequence* $\{t_n\} \subset \mathbb{R}$ *there can be extract a subsequence* $\{t_{k_n}\}$ *such that the sequence* $\{\varphi^{(t_{k_n})}\}$ *uniformly converges in the space* $L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu)$*, that is, there exists a function* $\widetilde{\varphi} \in L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu)$ *such that*

$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} \int_t^{t+1} |\varphi(s+t_{k_n}) - \widetilde{\varphi}(s)|^p ds = 0. \tag{1.109}$$

Remark 1.70. If the space $\mathfrak{B}$ is finite-dimensional, then the stability in the sense of Lagrange of the function $\varphi \in L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu)$ is equivalent to the following two conditions:

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} |\varphi(s)|^p ds < +\infty, \qquad \lim_{h\to 0} \sup_{t\in\mathbb{R}} \int_t^{t+1} |\varphi(s+h) - \varphi(s)|^p ds = 0. \tag{1.110}$$

Theorem 1.6.2. *Let* $\varphi \in L^p_{\text{loc}}(\mathbb{R}_+;\mathfrak{B};\mu)$*. The following statements are equivalent:*

(1) *the function* φ *is asymptotically* S^p *almost periodic, that is, the motion* $\sigma(\cdot,\varphi)$ *is asymptotically almost periodic in the dynamical system* $(L^p_{\text{loc}}(\mathbb{R}_+;\mathfrak{B};\mu),\mathbb{R},\sigma)$*;*

(2) *there exist an* S^p *almost periodic function* p *and a function* $\omega \in L^p_{\text{loc}}(\mathbb{R}_+;\mathfrak{B};\mu)$ *such that* $p \in L^p_{\text{loc}}(\mathbb{R};\mathfrak{B};\mu)$*,* $\varphi = p + \omega$ *and* $\lim_{t+\infty} \int_t^{t+1} |\omega(s)|^p ds = 0$*;*

(3) *the function* φ *is st.* L^+ *and* Σ^+_φ *is un. st.* $\mathcal{L}^+\Sigma^+_\varphi$ *in the dynamical system* $(L^p_{\text{loc}}(\mathbb{R}_+;\mathfrak{B};\mu),\mathbb{R},\sigma)$*;*

(4) *for every* $\varepsilon > 0$ *there exist numbers* $\beta \geq 0$ *and* $l > 0$ *such that on every segment of length* l *there is a number* τ *for which*

$$\int_t^{t+1} |\varphi(\tau+s) - \varphi(s)|^p ds < \varepsilon^p \tag{1.111}$$

for all $t \geq \beta$ *and* $t+\tau \geq \beta$*;*

(5) *from every sequence* $\{t_n\}$*,* $t_n \to +\infty$*, there can be extract a subsequence* $\{t_{k_n}\}$ *such that the sequence* $\{\varphi^{(t_{k_n})}\}$ *converges uniformly with respect to* $t \in \mathbb{R}_+$ *in the space* $L^p_{\text{loc}}(\mathbb{R}_+;\mathfrak{B};\mu)$*, that is, there exists a function* $\widetilde{\varphi} \in L^p_{\text{loc}}(\mathbb{R}_+;\mathfrak{B};\mu)$ *such that*

$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}_+} \int_t^{t+1} |\varphi(s+t_{k_n}) - \widetilde{\varphi}(s)|^p ds = 0. \tag{1.112}$$

2 Asymptotically Almost Periodic Solutions of Operator Equations

In this chapter we introduce the notion of comparability of motions of dynamical system by the character of their recurrence under limit. While studying asymptotically stable in the sense of Poisson motions this notion plays the same role that the notion of comparability by the character of recurrence of stable in the sense of Poisson motions introduced by B. A. Shcherbakov (see, e.g., [92, 100]).

2.1. Comparability of Motions by the Character of Recurrence

Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be dynamical systems, $x \in X$ and $y \in Y$. Denote by $\mathfrak{L}_{x,p}^{+\infty}$ the set of all sequences $\{t_n\} \in \mathfrak{M}_{x,p}$ such that $t_n \to +\infty$. Assume $\mathfrak{L}_x^{+\infty}(M) := \cup\{\mathfrak{L}_{x,p} : p \in M\}$ and $\mathfrak{L}_x^{+\infty} = \mathfrak{L}_x^{+\infty}(X)$.

Definition 2.1. A point $x \in X$ is called comparable by the character of recurrence with $y \in Y$ with respect to $M \subset Y$ or, in short, comparable with y with respect to the set M if $\mathfrak{L}_y^{+\infty}(M) \subseteq \mathfrak{L}_x^{+\infty}$.

Denote by $H(M) := \overline{\{\pi(t,x) : x \in M, t \in \mathbb{T}\}}$. Let $(Y, \mathbb{S}, \sigma)$ be a group dynamical system.

Lemma 2.2. *If* $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\infty}$, *then* $\mathfrak{L}_{y,\sigma(t,q)}^{+\infty} \subseteq \mathfrak{L}_{x,\pi(t,p)}^{+\infty}$ *for all* $t \in \mathbb{T} \subseteq \mathbb{S}$.

Proof. Let $t \in \mathbb{T}$ and $\{t_n\} \in \mathfrak{L}_{y,\sigma(t,q)}^{+\infty}$, then $t_n \to +\infty$ and $\sigma(t_n, y) \to \sigma(t,q)$ as $n \to +\infty$ and, consequently, $\{t_n - t\} \subset \mathbb{T}$ and $\{t_n - t\} \in \mathfrak{L}_{y,q}^{+\infty}$. In fact, $\lim_{n\to+\infty} \sigma(t_n - t, y) = \sigma(-t, \lim_{n\to+\infty} \sigma(t_n, y)) = \sigma(-t, \sigma(q,t)) = q$. So, $\{t_n - t\} \in \mathfrak{L}_{x,p}^{+\infty} \subseteq \mathfrak{L}_{y,p}^{+\infty}$, and hence $\{t_n - t\} \in \mathfrak{L}_{x,p}^{+\infty}$. Repeating the reasoning above it is easy to show that $\{t_n\} \in \mathfrak{L}_{x,\pi(t,p)}^{+\infty}$. The lemma is proved. □

Corollary 2.3. *Under the conditions of Lemma 2.2 if* $\mathfrak{L}_y^{+\infty}(M) \subseteq \mathfrak{L}_x^{+\infty}$, *then* $\mathfrak{L}_y^{+\infty}(\Sigma_M) \subset \mathfrak{L}_x^{+\infty}$, *where* $\Sigma_M := \{\pi(x,t) : x \in M,\ t \in \mathbb{T}\}$.

Lemma 2.4. *If* $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$, *then there exists a unique point* $p \in \omega_x$ *such that* $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\in fty}$.

Proof. Let $\{t_n\} \in \mathfrak{L}_{y,q}^{+\infty}$, then $yt_n \to q$. According to the conditions of the lemma there exists a point $p \in \omega_x$ such that $xt_n \to p$. Let us show that $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\in fty}$. Suppose that there exists $\{t'_n\} \in \mathfrak{L}_{y,q} \setminus \mathfrak{L}_{x,p}$, then there is a point $\overline{p} \in \omega_x$ $(\overline{p} \neq p)$ such that $\{xt'_n\}$ converges to $\overline{p}$. Let us compose the sequence $\{\overline{t}_k\}$ by the following rule:

$$\overline{t}_k = \begin{cases} t_n, & \text{if } k = 2n-1 \\ t'_n, & \text{if } k = 2n. \end{cases} \tag{2.1}$$

From the definition of the sequence $\{\overline{t}_k\}$ it follows that $\overline{t}_k \to +\infty$ and $y\overline{t}_k \to q$. Under the conditions of Lemma 2.4 $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$ and, consequently, $\{\overline{t}_k\} \in \mathfrak{L}_x^{+\infty}$, that is, $\{x\overline{t}_k\}$ is convergent. On the other hand, it has two different limit points p and $\overline{p}$. The obtained contradiction proves the inclusion $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\infty}$. To complete the proof of Lemma 2.4 it is sufficient to note that in the lemma the point p is uniquely defined because for two different points p_1 and p_2 there takes place the equality $\mathfrak{L}_{x,p_1}^{+\infty} \cap \mathfrak{L}_{x,p_2}^{+\infty} = \emptyset$. □

Theorem 2.1.1. *If a point x is comparable with y with respect to the set M, then there exists a continuous mapping $h : \sigma(\Sigma_M, \mathbb{T}) \to \omega_x$ satisfying the condition*

$$h(\sigma(q,t)) = \pi(h(q),t) \tag{2.2}$$

for all $q \in \sigma(\Sigma_M, \mathbb{T})$ and $t \in \mathbb{T}$.

Proof. Let the point x be comparable with y with respect to the set M. According to Corollary 2.3 $\mathfrak{L}_y^{+\infty}(\Sigma_M) \subseteq \mathfrak{L}_x^{+\infty}$. Let $q \in \Sigma_M$. By Lemma 2.4 there exists a single point $p \in \omega_x$ such that $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\infty}$. Assume $h(p) = q$. So, the mapping $h : \Sigma_M \to \omega_x$ is well defined. From Lemma 2.2, it follows that h satisfies (2.2). Let us show that the mapping h is continuous. Let $\{q_k\} \to q$ $(q_k, q \in \Sigma_M)$. Show that $\{p_k\} = \{h(q_k)\}$ converges to $p = h(q)$. For every $k \in \mathbb{N}$ choose $\{t_n^{(k)}\} \in \mathfrak{L}_{y,q_k}^{+\infty}$, then $p_k = h(q_k) = \lim_{n\to+\infty} xt_n^{(k)}$. Let $\varepsilon_k \downarrow 0$. For every $k \in \mathbb{N}$ we will choose $n_k \in \mathbb{N}$ such that the following inequalities would fulfill simultaneously

$$\rho(xt_{n_k}^{(k)}, p_k) < \varepsilon_k, \qquad d(yt_{n_k}^{(k)}, q_k) < \varepsilon_k \tag{2.3}$$

(it is clear that such n_k exist). Assume $t'_k := t_{n_k}^{(k)}$ and let us show that the sequence $\{t'_k\}$ belongs to $\mathfrak{L}_{y,q}^{+\infty}$. For this aim we will note that

$$d(yt'_k, q) \le d(yt'_k, q_k) + d(q_k, q) < \varepsilon_k + d(q_k, q). \tag{2.4}$$

Passing to limit in (2.4) as $k \to +\infty$ we will obtain $\{t'_k\} \in \mathfrak{L}_{y,q}^{+\infty}$. Since $\mathfrak{L}_{y,q}^{+\infty} \subseteq \mathfrak{L}_{x,p}^{+\infty}$, then $\{t'_k\} \in \mathfrak{L}_{x,p}$. As

$$\rho(p_k, p) \le \rho(p_k, xt'_k) + \rho(xt'_k, p) < \varepsilon_k + \rho(xt'_k, p), \tag{2.5}$$

then passing to limit in (2.5) and taking into consideration that $\{t'_k\} \in \mathfrak{L}_{x,p}^{+\infty}$ we will obtain $p_k \to p$. The theorem is proved. □

Let a point x be comparable with y with respect to M. Note that at the point of view of applications (see, e.g., [92, 93, 100]) the following cases are the most important.

(1) $\mathfrak{L}_{y,y}^{+\infty} \subseteq \mathfrak{L}_{x,x}^{+\infty}$.

As it is shown in [100, 105], the inclusion $\mathfrak{L}_{y,y}^{+\infty} \subseteq \mathfrak{L}_{x,x}^{+\infty}$ takes place if and only if $\mathfrak{N}_y \subseteq \mathfrak{N}_x$. As it was mentioned in Section 1.2 of Chapter 1, the inclusion $\mathfrak{N}_y \subseteq \mathfrak{N}_x$ takes place if and only if x is comparable by recurrence with y.

(2) $\mathfrak{L}_{y}^{+\infty} \subseteq \mathfrak{L}_{x}^{+\infty}$ and $\mathfrak{L}_{y,y}^{+\infty} \subseteq \mathfrak{L}_{x,x}^{+\infty}$.

Assume $\mathfrak{M}_y^{+} = \{\{t_n\} : \{t_n\} \in \mathfrak{M}_y,\ t_n \in \mathbb{T}_+\}$.

Definition 2.5. One will call the point x strongly comparable (in the positive direction) with y if $\mathfrak{L}_{y,y}^{+\infty} \subseteq \mathfrak{L}_{x,x}^{+\infty}$ and $\mathfrak{L}_{y}^{+\infty} \subseteq \mathfrak{L}_{x}^{+\infty}$.

The next theorem takes place.

Theorem 2.1.2. *The following statements are equivalent:*

(1) *the point x is strongly comparable with y;*

(2) *there exists a continuous mapping $h : H^+(y) \to H^+(x)$ satisfying the condition (2.2) for all $q \in H^+(y)$ and $t \in \mathbb{T}_+$, and besides $h(y) = x$;*

(3) $\mathfrak{M}_y^{+} \subseteq \mathfrak{M}_x^{+}$.

Proof. Let us show that from (1) it follows (2). In fact, according to Theorem 2.1.1 there exists a continuous mapping $h : H^+(y) \to H^+(x)$ with the properties needed. Suppose that the condition (2) is fulfilled. And let $\{t_n\} \in \mathfrak{M}_y^{+}$. Then there exists a point $q \in H^+(y)$ such that $yt_n \to q$. In virtue of the condition

$$\{h(yt_n)\} = \{h(y)t_n\} = \{xt_n\} \longrightarrow h(q) \tag{2.6}$$

and, consequently, $\{t_n\} \in \mathfrak{M}_x^{+}$. At last, we will show from (3) it follows (1). It is clear that to prove (1) it is sufficient to see that $\mathfrak{L}_{y,y}^{+\infty} \subseteq \mathfrak{L}_{x,x}^{+\infty}$. If we suppose that the inclusion $\mathfrak{L}_{y,y}^{+\infty} \subseteq \mathfrak{L}_{x,x}^{+\infty}$, does not take place, then there exists $\{\bar{t}_n\} \in \mathfrak{L}_{y,y}^{+\infty} \setminus \mathfrak{L}_{x,x}^{+\infty}$. Since $\mathfrak{L}_{y,y}^{+\infty} \subseteq \mathfrak{L}_{y}^{+\infty} \subseteq \mathfrak{L}_{x}^{+\infty}$, there exists a point $p \neq x$ such that $\{\bar{t}_n\} \in \mathfrak{L}_{x,p}^{+\infty}$. Let us compose the sequence $\{t_k'\}$ by the following condition:

$$t_k' = \begin{cases} \bar{t}_n, & \text{if } k = 2n-1 \\ t_n', & \text{if } k = 2n \end{cases} \tag{2.7}$$

for every $k \in \mathbb{N}$. It is easy to see that $\{t_k'\} \in \mathfrak{M}_{y,y}^{+}$ and, consequently, $\{t_k'\} \in \mathfrak{M}_x^{+}$. So, the sequence $\{xt_k'\}$ is convergent. From this fact it follows that $x = p$. The last equality contradicts to the choice of p $(p \neq x)$. The theorem is proved. □

Remark 2.6. From Theorem 2.1.2 and from the results of the works [100, 105] it follows that the strong comparability of the point x with y is equivalent to their uniform comparability if the point y is st. L^+. In general case these notions are apparently different (though we do not know the according example).

(3) $\mathfrak{L}_{y}^{+\infty} \subseteq \mathfrak{L}_{x}^{+\infty}$.

Definition 2.7. One will say that the point x is comparable in limit (in the positive direction) with the point y if $\mathfrak{L}_y^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$.

2.2. Comparability in Limit of Asymptotically Poisson Stable Motions

Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be dynamical systems, $x \in X$, and $y \in Y$.

Theorem 2.2.1. *Let the point y be asymptotically Poisson stable. The point x is comparable in limit with y if and only if there exists a continuous mapping $h : \omega_y \to \omega_x$ satisfying the condition*

$$h(\sigma(t,q)) = \pi(t, h(q)) \tag{2.8}$$

for all $q \in \omega_y$, $t \in \mathbb{T}$ and

$$\lim_{n\to+\infty} \rho(\pi(t_n,x), \pi(t_n, h(\tilde{q}))) = 0 \tag{2.9}$$

for all $\tilde{q} \in P_y$ and $\{t_n\} \in \mathfrak{L}_y^{+\infty}$.

Proof. Necessity. Let the point x be comparable in limit with y. According to Theorem 2.1.1 there exists a continuous mapping $h : \omega_y \to \omega_x$ satisfying (2.8). Let us show that (2.9) takes place too. Let $\tilde{q} \in P_y$ and $\{t_n\} \in \mathfrak{L}_y^{+\infty}$. Then there exists a point $\overline{q} \in \omega_y$ such that $yt_n \to \overline{q}$. By the definition of the mapping h we have $h(\overline{q}) = \lim_{n\to+\infty} xt_n$. On the other hand, $\overline{q} = \lim_{n\to+\infty} yt_n = \lim_{n\to+\infty} \tilde{q}t_n$ as $\tilde{q} \in P_y$. Hence, $h(\overline{q}) = h(\lim_{n\to+\infty} yt_n) = h(\lim_{n\to+\infty} \tilde{q}t_n) = \lim_{n\to+\infty} \pi(t_n, h(\tilde{q}))$. So, $\lim_{n\to+\infty} xt_n = h(\tilde{q}) = \lim_{n\to+\infty} \pi(t_n, h(\tilde{q}))$. From this it follows (2.9).

Sufficiency. Let exist a continuous mapping $h : \omega_y \to \omega_x$ satisfying (2.8) and (2.9). Let us take an arbitrary sequence $\{t_n\} \in \mathfrak{L}_y^{+\infty}$, then there exists a point $q \in \omega_y$ such that the sequence $\{yt_n\}$ converges to q. Let $y \in P_y$. Note that

$$h(q) = h\Big(\lim_{n\to+\infty} yt_n\Big) = h\Big(\lim_{n\to+\infty} \sigma(t_n, q)\Big) = \lim_{n\to+\infty} \pi(t_n, h(q)) \tag{2.10}$$

and, consequently,

$$\rho(xt_n, h(q)) \le \rho(xt_n, h(q)t_n) + \rho(h(q)t_n, h(q)). \tag{2.11}$$

Passing to limit in (2.11) and taking into consideration (2.10) and (2.9), we get the equality $h(q) = \lim_{n\to+\infty} xt_n$, that is, $\{t_n\} \in \mathfrak{L}_x^{+\infty}$ and $\mathfrak{L}_y^{+\infty} \subset \mathfrak{L}_x^{+\infty}$. The theorem is proved. □

Corollary 2.8. *Let the point y be st. L^+ and asymptotically Poisson stable. The point x is comparable in limit if and only if there exists a continuous mapping $h : \omega_y \to \omega_x$ satisfying* (2.8) *and*

$$\lim_{t\to+\infty} \rho(xt, h(q)t) = 0 \tag{2.12}$$

for all $q \in P_y$.

Proof. The sufficiency of Corollary 2.8 it follows from Theorem 2.2.1. Let us prove the necessity. Let the point x be comparable in limit with y. According to Theorem 2.2.1 there exists a continuous mapping $h : \omega_y \to \omega_x$ satisfying (2.8) and (2.9). Suppose that (2.12) does not take place. Then there exist $q \in P_y$, $t_n \to +\infty$, and $\varepsilon_0 > 0$ such that

$$\rho(xt_n, h(q)t_n) \geq \varepsilon_0. \tag{2.13}$$

By st. L^+ of the point y from the sequence $\{t_n\}$ we can choose a subsequence $\{t_{k_n}\} \in \mathfrak{L}_y^{+\infty}$. According to Theorem 2.2.1 the equality

$$\lim_{n\to+\infty} \rho(xt_{k_n}, h(q)t_{k_n}) = 0 \tag{2.14}$$

is held. Passing to limit with respect to the subsequence $\{t_{k_n}\}$ in (2.13) and taking into consideration (2.14), we obtain $\varepsilon_0 \leq 0$. The last contradicts to the choice of the number ε_0. So, the obtained contradiction proves our statement. □

The theorem given below shows that the introduced notion of comparability in limit plays the same role while studying asymptotically Poisson stable motions as it does the notion of comparability in the sense of Shcherbakov for Poisson stable motions [92, 100].

Theorem 2.2.2. *Let y be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent). If the point x is comparable in limit with y, then the point x is also asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).*

Proof. Let x be comparable in limit with y. According to Corollary 2.8 there exists a continuous mapping $h : \omega_y \to \omega_x$ satisfying (2.8) and (2.12). As y is st. L^+, then ω_y is compact and, consequently, the mapping h is uniformly continuous. Let $q \in P_y$, then $p = h(q) \in P_x$ and by [105, Theorem 9] the point p is stationary (resp., τ-periodic, almost periodic, recurrent). The theorem is proved. □

2.3. Asymptotically Poisson Stable Solutions

Let us consider the problem of existence of asymptotically Poisson stable solutions for operator equations.

Let $h : X \to Y$ be a homomorphism of the system $(X, \mathbb{T}, \pi)$ onto $(Y, \mathbb{T}, \sigma)$.

Consider the operator equation

$$h(x) = y, \tag{2.15}$$

where $y \in Y$. Along with (2.15) we will consider the family of "ω-limit" equations

$$h(x) = q, \quad (q \in \omega_y). \tag{2.16}$$

Theorem 2.3.1. *If a solution x of* (2.15) *is st. L^+ and every* (2.16) *admits at most one solution from ω_x, then x is comparable in limit with $y \in Y$.*

Proof. Let $\{t_n\} \in \mathfrak{L}_y^{+\infty}$. Then there exists a point $q \in \omega_y$ such that $yt_n \to q$. In virtue of st. L^+ of the solution x the sequence $\{xt_n\}$ is relatively compact. Let p be an arbitrary limit point of the sequence $\{xt_n\}$, then there exists a subsequence $\{t_{k_n}\} \subseteq \{t_n\}$ such that the sequence $\{xt_{k_n}\}$ converges to p. Since h is continuous and homomorphic, there takes place the equality $h(p) = q$. So, p is a solution of (2.16), $p \in \omega_x$ and, consequently, every limit point p of the sequence $\{xt_n\}$ is a solution of (2.16). Under the conditions of Theorem 2.3.1, (2.16) has at most one solution from ω_x. Hence, the sequence $\{xt_n\}$ has exactly one limit point. As the sequence $\{xt_n\}$ is relatively compact, it converges. So, the sequence $\{t_n\} \in \mathfrak{L}_x^{+\infty}$ and we proved the inclusion $\mathfrak{L}_y^{+\infty} \subset \mathfrak{L}_x^{+\infty}$. □

Corollary 2.9. *Let x be st. L^+ solution of* (2.15) *and y be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent). If every* (2.16) *admits at most one solution from ω_x, then x is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).*

Proof. The formulated statement follows from Theorems 2.2.2 and 2.3.1. □

Definition 2.10. A solution $x \in M$ ($M \subseteq X$) of (2.15) is called separated in the set M if x is a unique solution of (2.15) from M or there exists a number $r > 0$ such that whatever would be a solution $p \in M$ ($p \neq x$) of (2.15), $\rho(xt, pt) \geq r$ for all $t \in \mathbb{T}$.

Lemma 2.11. *Let a set $M \subseteq X$ be a compact set. If every solution $x \in M$ of* (2.15) *is separated in M, then* (2.15) *has finite number of solutions from M.*

Proof. Suppose the contrary, that is, in the set M there exists the infinite set $\{x_n\}$ of different solutions of (2.15). By compactness of M we can extract a convergent subsequence from the sequence $\{x_n\}$. Without loss of generality we can consider that $\{x_n\}$ is convergent. Let $p = \lim_{n\to+\infty} x_n$. Since h is continuous and M is closed, $h(p) = y$ and $p \in M$. So, p is a solution of (2.15), but obviously it is not separated in M that contradicts to the condition. The lemma is proved. □

Lemma 2.12. *Let a point $y \in Y$ be recurrent, $M \subseteq X$ be a compact invariant set, and $y \in h(M)$. If solutions from M of every* (2.16) *are separated in the set M, then there exists a number $r > 0$ such that $\rho(p_1 t, p_2 t) \geq r > 0$ for all $t \in \mathbb{T}$ and $p_1, p_2 \in M$ with $h(p_1) = h(p_2)$ and $p_1 \neq p_2$.*

Proof. By Lemma 2.11 for every $q \in \omega_y$ (2.16) has finite number of solutions from M. Denote by $n(q)$ the number of different solutions of (2.16) from M. Let us show that the number $n(q)$ does not depend on the point $q \in \omega_y$. In fact, for the point $q \in \omega_y$ there exists a sequence $\{t_n\} \in \mathfrak{L}_y^{+\infty}$ such that the sequence $\{yt_n\}$ converges to q. Consider the sequence $\{\xi_n\} \subseteq M^M$ defined by the equality $\xi_n(x) = \pi(x, t_n)$ for all $x \in M$. According to theorem of Tikhonoff [106] the sequence $\{\xi_n\}$ is relatively compact in M^M. Without loss of generality we can consider that $\{\xi_n\}$ converges in M^M. Assume $\xi := \lim_{n\to+\infty} \xi_n$. Denote by $x_1, x_2, \ldots, x_{n(y)}$ solutions of (2.15) that are from M and $\overline{x}_i := \xi(x_i)$ for all $i = 1, 2, \ldots, n(y)$, that is, $\overline{x}_i = \lim_{n\to+\infty} x_i t_n$. Since h is continuous and homomorphic, the points $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{n(y)}$ are the solutions of (2.16). Let us show that the points $\overline{x}_i$ ($i = 1, 2, \ldots, n(y)$) are different. As $\xi(x) = \lim_{n\to+\infty} \xi_n(x)$ for all $x \in M$ and M is invariant, then, in particular,

$$\xi(\pi(t,x_i)) = \lim_{n\to+\infty} \pi(t_n,\pi(t,x_i)) = \lim_{n\to+\infty} \pi(t,\pi(t_n,x_i)) = \pi(t,\overline{x}_i). \tag{2.17}$$

Assume $r := \inf\{\rho(\pi(x_i,t),\pi(x_j,t)) : i \neq j,\ t \in \mathbb{T}\}$. Then under the conditions of Lemma 2.12 $r > 0$. Since $\rho(\pi(x_i,t),\pi(x_j,t)) \geq r > 0$ for all $t \in \mathbb{T}$ and $i \neq j$ $(1 \leq i, j \leq n(y))$, then the inequality

$$\rho(\pi(t+t_n,x_i),\pi(t+t_n,x_j)) \geq r \tag{2.18}$$

is fulfilled. Passing to limit in inequality (2.18) and taking into consideration (2.17), we obtain

$$\rho(\pi(\overline{x}_i,t),\pi(\overline{x}_j,t)) \geq r \tag{2.19}$$

for all $t \in \mathbb{T}$ and $i \neq j$ $(1 \leq i, j \leq n(y))$. From (2.19) it follows that $\overline{x}_i$ are different. So, $n(q) \geq n(y)$. From the recurrence of y it follows that $y \in \omega_q$. Repeating the reasoning above we obtain the inequality $n(y) \geq n(q)$. Hence, $n(q) = n(y)$ for all $q \in \omega_y$. From (2.19) it follows that the number $r > 0$ possesses the needed properties. □

Lemma 2.13. *Let a point $y \in Y$ be asymptotically recurrent and x be a st. L^+ solution of* (2.15). *If for every $q \in \omega_y$ all the solutions from ω_x of* (2.16) *are separated in ω_x, then there exists a unique solution $p \in \omega_x$ of* (2.16) *such that $p \in P_x$.*

Proof. According to Lemma 2.11, under the conditions of Lemma 2.13, (2.16) has a finite number n of solutions $p_1, p_2, \ldots, p_n$ from ω_x. Let us show that

$$\lim_{t\to+\infty} \inf\{\rho(xt,p_it) : 1 \leq i \leq n\} = 0. \tag{2.20}$$

Suppose the contrary. Then there exist $\varepsilon_0 > 0$ and $\{t_n\} \to +\infty$ such that

$$\rho(xt_k,p_it_k) \geq \varepsilon_0 \tag{2.21}$$

for all $i = 1,2,\ldots,n$ and $k = 1,2,\ldots$. Since the point x is st. L^+ and y is asymptotically recurrent, the sequences $\{xt_k\}$, $\{p_it_k\}$ $(i = 1,2,\ldots,n)$, and $\{yt_k\}$ can be considered convergent. Assume $\overline{p} := \lim_{k\to+\infty} xt_k$, $\overline{q} := \lim_{k\to+\infty} yt_k$, and $\overline{p}_i := \lim_{k\to+\infty} p_it_k$. From (2.21) it follows that $\overline{p} \neq \overline{p}_i$ $(i = \overline{1,n})$. On the other hand, $\overline{p} \in \omega_x$, $h(\overline{p}) = \overline{q}$, and by Lemma 2.12 $X_{\overline{q}} \cap \omega_x = \{\overline{p}_1,\overline{p}_2,\ldots,\overline{p}_n\}$, where $X_{\overline{q}} = h^{-1}(\overline{q})$. So, $\overline{p} \in \{\overline{p}_1,\overline{p}_2,\ldots,\overline{p}_n\}$. The last inclusion contradicts to the condition that $\overline{p}_i \neq \overline{p}$ $(i = 1,2,\ldots,n)$. So, (2.20) is proved.

Let us show that there exists a number $1 \leq i_0 \leq n$ for which $p_{i_0} \in P_x$, that is,

$$\lim_{t\to+\infty} \rho(xt,p_{i_0}t) = 0. \tag{2.22}$$

For a number ε, $0 < \varepsilon < r/3$, ($r > 0$ is the number the existence of which is guaranteed by Lemma 2.12) we will find $L(\varepsilon) > 0$ such that

$$\inf\{\rho(xt,p_it) : 1 \leq i \leq n\} < \varepsilon \tag{2.23}$$

for all $t \geq L(\varepsilon)$. Let $t_0 > L(\varepsilon)$, then there exists $1 \leq i_1 \leq n$ such that

$$\rho(xt_0, p_{i_1}t_0) < \varepsilon. \tag{2.24}$$

Assume $\delta(t_0) := \sup\{\widetilde{\delta} : \rho(xt, p_{i_1}t) < \varepsilon \text{ for all } t \in [t_0, t_0 + \widetilde{\delta}]\}$. Let us show that $\delta(t_0) = +\infty$. Suppose the contrary, then

$$\rho(xt_0', p_{i_1}t_0') \geq \varepsilon, \tag{2.25}$$

where $t_0' = t_0 + \delta(t_0)$, and there exists $i_2 \neq i_1$ $(1 \leq i_2 \leq n)$ such that

$$\rho(xt_0', p_{i_2}t_0') < \varepsilon. \tag{2.26}$$

On the other hand,

$$\rho(xt_0', p_{i_2}t_0') \geq \rho(p_{i_2}t_0', p_{i_1}t_0') - \rho(p_{i_1}t_0', xt_0') > r - \varepsilon > 2\varepsilon. \tag{2.27}$$

Inequality (2.27) contradicts to the assumption. So, we found $L(\varepsilon) > 0$ and $p_{i_0} \in \{p_1, p_2, \ldots, p_n\}$ such that

$$\rho(xt, p_{i_0}t) < \varepsilon \tag{2.28}$$

for all $t \geq L(\varepsilon)$. Assume $p := p_{i_0}$ and let us show that the point p does not depend on the choice of ε. In fact, if we suppose the contrary, then we can find numbers ε_1 and ε_2, points p_1 and p_2 $(p_1 \neq p_2)$, and $L(\varepsilon_1) > 0$ and $L(\varepsilon_2) > 0$ satisfying the conditions mentioned above. Assume $L := \max(L(\varepsilon_1), L(\varepsilon_2))$, then

$$\rho(p_1t, p_2t) \leq \rho(p_1t, xt) + \rho(xt, p_2t) \leq \varepsilon_1 + \varepsilon_2 < \frac{2r}{3} < r. \tag{2.29}$$

Inequality (2.29) contradicts to the choice of r (see Lemma 2.12). The lemma is proved. □

Theorem 2.3.2. *Let a point $y \in Y$ be asymptotically almost periodic (resp., asymptotically recurrent) and let x be a st. L^+ solution of* (2.15). *If for every $q \in \omega_y$ all the solutions from ω_x of* (2.16) *are separated in ω_x, then x is asymptotically almost periodic (resp., asymptotically recurrent).*

Proof. According to Lemma 2.13 there exists a unique point $p \in \omega_x$ such that $p \in P_x$. To complete the proof of Theorem 2.3.2 it remains to note that under the conditions of Theorem 2.3.2 the set ω_x consists of almost periodic (resp., recurrent) motions. The last statement it follows from [107, Theorem 3, page 111] (the case of $\mathbb{T} = \mathbb{Z}$ see in [94]) and [93, Theorem 14.7]. The theorem is proved. □

Definition 2.14. One will say that a solution x of (2.15) is Σ^+-stable if for every $\varepsilon > 0$, one can find $\delta > 0$ such that if $\rho(xt_1, xt_2) < \delta$ and

$$\sup\{d(y(t+t_1), y(t+t_2)) : t \in \mathbb{T}_+\} < \delta \quad (t_1, t_2 \in \mathbb{T}_+), \tag{2.30}$$

then

$$\sup\{\rho(x(t+t_1),x(t+t_2)) : t\in\mathbb{T}_+\} < \varepsilon. \tag{2.31}$$

Theorem 2.3.3. *Let a point y be asymptotically almost periodic and a point x be st. L^+. If x is a Σ^+-stable solution of* (2.15)*, then it is asymptotically almost periodic.*

Proof. Let x be a solution of (2.15) satisfying the conditions of Theorem 2.3.3 and $\varepsilon > 0$. Choose $\delta > 0$ out of the condition of Σ^+-stability of x. From Theorem 1.3.2 it follows that to show that the point x is asymptotically almost periodic it is sufficient to show that from every sequence $\{t_n\} \to +\infty$ we can extract a subsequence $\{t_{k_n}\}$ such that $\{xt_{k_n}\}$ converges uniformly with respect to $t\in\mathbb{T}_+$.

Let $\{t_n\} \to +\infty$. In virtue of the statements that we have done concerning x and y, the sequences $\{xt_n\}$ and $\{yt_n\}$ can be considered convergent, moreover, the second one can be considered uniformly with respect to $t\in\mathbb{T}_+$. So, there is a number $n_0\in\mathbb{N}$ such that

$$\sup\{d(y(t+t_n),y(t+t_m)) : t\in\mathbb{T}_+\} < \delta, \tag{2.32}$$

$$\rho(xt_n,xt_m) < \delta \tag{2.33}$$

for all $m,n \geq n_0$. By the choice of the number δ from inequalities (2.32) and (2.33) it follows that for $m,n\geq n_0$

$$\sup\{\rho(x(t+t_n),x(t+t_m)) : t\in\mathbb{T}_+\} < \varepsilon. \tag{2.34}$$

From (2.34) and the completeness of the space X it follows that $\{xt_n\}$ converges uniformly with respect to $t\in\mathbb{T}_+$. The theorem is proved. □

Theorem 2.3.4. *If a point $y\in Y$ is τ-periodic, x is a st. L^+ solution of* (2.15) *and the set $M = \{\pi(n\tau,x) : n\in\mathbb{N}\}$ is un. st. $\mathcal{L}^+M$, then the solution x is asymptotically almost periodic.*

Proof. Let us consider a cascade $(X_y,\overline{\pi})$ generated by positive powers of the mapping $\overline{\pi} : X_y \to X_y$, where $X_y = h^{-1}(y)$ and $\overline{\pi}(z) := \pi(\tau,z)$ for all $z\in X_y$. Note that the point $x\in X_y$ is st. L^+ in discrete dynamical system $(X_y,\overline{\pi})$ too. Besides, under the conditions of the theorem the positive semitrajectory $\{\pi(n\tau,x) : n\in\mathbb{Z}_+\}$ of $x\in X_y$ in the dynamical system $(X_y,\overline{\pi})$ is un. st. $\mathcal{L}^+$ with respect to itself and, according to Theorem 1.3.2, the point x is asymptotically almost periodic in the dynamical system $(X_y,\overline{\pi})$, that is, there exists an almost periodic in $(X_y,\overline{\pi})$ point $p\in X_y$ such that

$$\lim_{k\to+\infty}\rho(\pi(k\tau,x),\pi(k\tau,p)) = 0. \tag{2.35}$$

Further let us show that from (2.35) it follows (1.4). Suppose the contrary. Then there exists $\{t_n\}\subseteq\mathbb{T}_+$ ($t_n\to+\infty$) and a positive number ε_0 such that

$$\rho(xt_n,pt_n) \geq \varepsilon_0. \tag{2.36}$$

Denote by k_n the integer part of t_n after the division by τ. Then $t_n = k_n\tau + \bar{t}_n$, where $\bar{t}_n \in [0,\tau[$ and, consequently, $\{\bar{t}_n\}$ can be considered convergent. Assume $\bar{t} := \lim_{n\to+\infty}\bar{t}_n$. Since the point x is st. L^+ and also taking into consideration (2.35), we can consider that the sequences $\{\pi(k_n\tau, x)\}$ and $\{\pi(k_n\tau, p)\}$ converge to the same point $\bar{p}$. Note that

$$\varepsilon_0 \le \rho(\pi(t_n,x),\pi(t_n,p)) = \rho(\pi(k_n\tau+\bar{t}_n,x),\pi(k_n\tau+\bar{t}_n,p)) = \rho(\pi(\bar{t}_n,\pi(k_n\tau,x)),$$
$$\pi(\bar{t}_n,\pi(k_n\tau),\bar{p})) \le \rho(\pi(\bar{t}_n,\pi(k_n\tau,x)),\pi(\bar{t},\bar{p})) + \rho(\pi(\bar{t},\bar{p})),\pi(\bar{t}_n,\pi(k_n\tau,p)). \tag{2.37}$$

Passing to limit in (2.37) as $n \to +\infty$, we will obtain $\varepsilon_0 \le 0$. The last contradicts to the choice of the number ε_0. The necessary statement is proved. To finish the proof of the theorem is sufficient to show the point $p \in X_y$ is almost periodic in the dynamical system $(X,\mathbb{T},\pi)$. In fact, as we know [86, 93, 99], the point $p \in X$ is almost periodic if and only if from every sequence $\{t_n\} \subset \mathbb{T}$ we can extract a subsequence $\{t_{k_n}\}$ such that $\{\pi(t_{k_n},x)\}$ converges uniformly with respect to $t \in \mathbb{T}$. Let $\{t_n\} \subset \mathbb{T}$ be an arbitrary sequence. Then $t_n = l_n\tau + \bar{t}_n$ where $l_n \in \mathbb{Z}$ and $\bar{t}_n \in [0,\tau[$. The sequence $\{\bar{t}_n\}$ can be considered convergent. Assume $\bar{t} := \lim_{n\to+\infty}\bar{t}_n$. Since the point $p \in X_y$ is almost periodic in $(X_y,\bar{\pi})$, then from the sequence $\{l_n\}$ we can extract a subsequence $\{l_{k_n}\}$ such that

$$\lim_{n,m\to+\infty} \sup\{\rho(\pi(l_{k_n}\tau+s,p),\pi(l_{k_m}\tau+s,p)) : s \in \mathbb{Z}\} = 0. \tag{2.38}$$

From (2.38) and the uniform integral continuity on $\overline{\{\pi(p,t) : t \in \mathbb{T}\}} := H(p)$ it follows that

$$\lim_{n,m\to+\infty} \sup\{\rho(\pi(l_{k_n}\tau+s,p),\pi(l_{k_m}\tau+s,p)) : s \in \mathbb{T}\} = 0. \tag{2.39}$$

Taking into consideration the completeness of the space X and (2.39), we make the conclusion that the sequence $\{\pi(l_{k_n}\tau,p)\}$ converges uniformly with respect to $t \in \mathbb{T}$ and, hence, the sequence $\{\pi(t_{k_n},p)\}$ also converges uniformly on $\mathbb{T}$. The theorem is completely proved. □

2.4. Asymptotically Periodic Solutions

Theorem 2.4.1. *Let x be a st. L^+ solution of* (2.15) *with an asymptotically τ-periodic point y and $\bar{q} = \lim_{k\to+\infty}\sigma(k\tau,y)$. If the equation*

$$h(x) = \bar{q} \tag{2.40}$$

admits at most one solution from ω_x, then the solution x is asymptotically τ-periodic.

Proof. Let us prove that the sequence $\{\pi(k\tau,x)\}$ is convergent. Since x is st. L^+, then for the convergence of the sequence $\{\pi(k\tau,x)\}$ it is sufficient that it would contain at most one limit point. Let x_1 be x_2 be two arbitrary limit points of the sequence $\{\pi(k\tau,x)\}$. Then there exist sequences $\{k_n^i\}$ $(i = 1,2)$ such that $x_i = \lim_{n\to+\infty}\pi(k_n^i\tau,x)$ $(i = 1,2)$. Since $\{\sigma(k\tau,y)\} \to \bar{q}$, then $h(x_1) = h(x_2) = \bar{q}\; x_1, x_2 \in \omega_x$. According to the conditions of Theorem 2.4.1 $x_1 = x_2$ and, consequently, the sequence $\{\pi(k\tau,x)\}$ is convergent and by Theorem 1.4.1 the point x is asymptotically τ-periodic. □

Corollary 2.15. *Let x be a st. L^+ solution of* (2.15) *with the asymptotically stationary point y and $\overline{q} := \lim_{t\to+\infty}\sigma(t,y)$. If* (2.40) *admits at most one solution from ω_x, then the solution x is asymptotically stationary.*

Proof. The formulated statement it follows from Theorem 2.4.1 and Corollary 1.38. □

Theorem 2.4.2. *Let x be a st. L^+ solution of* (2.15), $\mathbb{T} = \mathbb{R}$ *or* $\mathbb{R}_+$, *and the set X_y be homeomorphic to $\mathbb{R}$. If the point y is τ-periodic, then the solution x is asymptotically τ-periodic.*

Proof. Let the point y be τ-periodic. Without loss of generality we can assume that $X_q = \mathbb{R}$ for every $q \in \omega_y$. According to Theorem 1.4.1 for the asymptotical τ-periodicity of the point x it is sufficient to show that $\{\pi(k\tau,x)\}$ is convergent. Let us consider the function $\varphi(t) = \pi(t+\tau,x) - \pi(t,x)$. Logically, two cases are possible:

(a) there exists $\overline{t} \in \mathbb{T}$ such that $\varphi(\overline{t}) = 0$ and, consequently, $\pi(\overline{t}+\tau,x) = \pi(\overline{t},x)$. From this it follows that $\pi(t+\tau,\overline{x}) = \pi(t,\overline{x})$ for all $t \in \mathbb{T}$, where $\overline{x} = \pi(\overline{t},x)$, and, consequently, x is asymptotically τ-periodic;

(b) the function $\varphi(t)$ keeps the sign. It is not difficult to see that in this case the sequence $\{\pi(k\tau,x)\}$ is monotone and, consequently, convergent. The theorem is proved. □

Theorem 2.4.3. *Let x be a st. L^+ solution of* (2.15) *with the asymptotically τ-periodic point y. If all solutions of* (2.40) *from ω_x are separated in ω_x, then the solution x is asymptotically $m_0\tau$-periodic, where m_0 is some integer number.*

Proof. According to Lemma 2.11, (2.40) has only finite number of solutions $p_1, p_2, \dots, p_{n_0}$ from ω_x. Let us consider a cascade $(X_y, \overline{\pi})$ generated by positive powers of $\overline{\pi} := \pi(\tau,\cdot) : X_y \to X_y$ ($\overline{\pi}(x) := \pi(x,\tau)$ for all $x \in X_y$). Since x is a st. L^+ solution of (2.15), then the trajectory $\{\overline{\pi}^k x \mid k \in \mathbb{N}\} = \{\pi(k\tau,x) \mid k \in \mathbb{N}\}$ of the point $x \in X$ is relatively compact in the dynamical system $(X_y,\overline{\pi})$. Note that every limit point of the sequence $\{\pi(k\tau,x)\}$ is a solution of (2.40) and belongs to ω_x and, in virtue of the said above, it is contained in the set $\{x_1, x_2, \dots, x_{n_0}\}$. Hence, in the dynamical system $(X_y,\overline{\pi})$ the ω-limit set $\overline{\omega}_x$ of the point x consists of finite number of points. Let $\overline{\omega}_x = \{\overline{x}_1, \overline{x}_2, \dots, \overline{x}_{m_0}\}$ ($m_0 \le n_0$). Then, according to Theorem 1.4.3, the point x is asymptotically m_0-periodic in the system $(X_y,\overline{\pi})$ and, consequently, the sequence $\{\pi(m_0 k\tau,x)\}$ is convergent. By Theorem 1.4.1 the point x is asymptotically $m_0\tau$-periodic in the system $(X,\mathbb{T},\pi)$. The theorem is proved. □

2.5. Homoclinic and Heteroclinic Motions

Everywhere in this section we will assume that $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$. Denote by $P(X)$ the set of all Poisson stable points of the dynamical system $(X,\mathbb{T},\pi)$, that is, $P(X) := \{x \mid x \in X, x \in \omega_x \cap \alpha_x\}$, where $\alpha_x := \bigcap_{t\le 0}\overline{\bigcup_{\tau\le t}\pi(x,\tau)}$.

Definition 2.16. A point $x \in X$ (or a motion $\pi(\cdot,x)$) is called heteroclinic, if there exist points $p_1, p_2 \in P(X)$ such that $x \in W^u(p_1) \cap W^s(p_2)$, where $W^s(p) := \{x \mid x \in X, \lim_{t\to+\infty}\rho(xt,pt) = 0\}$ and $W^u(p) := \{p \in X \mid \lim_{t\to-\infty}\rho(xt,pt) = 0\}$. If $p_1 = p_2 = p$, then the point x is called homoclinic.

Remark 2.17. (1) It is convenient to denote a heteroclinic (resp., homoclinic) point by $(x; p_1, p_2)$ $((x; p))$, where p_1 and p_2 $(p_1 = p_2 = p)$ are Poisson stable points figuring in the definition of the heteroclinic (resp., homoclinic) point x.

(2) Note, that the point x is heteroclinic, if it is asymptotically Poisson stable in positive and negative directions, that is, it is Poisson bistable.

(3) According to Corollary 1.29, if the points p_1 and p_2, figuring in the definition of the heteroclinic point $(x; p_1, p_2)$, are almost periodic, then they are uniquely defined. If they are recurrent, then, generally speaking, they are not (see, i.e., [93, page 157]).

Denote by $\mathfrak{L}_{x,p}^{\pm\infty} := \{\{t_n\} \mid \{t_n\} \in \mathfrak{L}_{x,p},\ t_n \to \pm\infty\}$, $\mathfrak{L}_x^{\pm\infty} := \cup\{\mathfrak{L}_{x,p}^{\pm\infty} \mid p \in X\}$, $\mathfrak{L}_x := \{\{t_n\} \mid |t_n| \to +\infty,\ \{t_n\} \in \mathfrak{M}_x\}$ and $\mathfrak{L}_x^{\infty} := \mathfrak{L}_x^{+\infty} \cup \mathfrak{L}_x^{-\infty}$.

Definition 2.18. Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be dynamical systems, $x \in X$, $y \in Y$. One will say that the point x is comparable in limit in positive (resp., negative) direction with respect to the character of recurrence with the point y, if $\mathfrak{L}_y^{+\infty} \subseteq \mathfrak{L}_x^{+\infty}$ (resp., $\mathfrak{L}_y^{-\infty} \subseteq \mathfrak{L}_x^{-\infty}$). If x is comparable in limit with y both in positive and negative direction, then we will say that x is comparable in limit with respect to the character of recurrence with y. At last, if $\mathfrak{L}_y \subseteq \mathfrak{L}_x$, then we will say that x is strongly comparable in limit with y.

Remark 2.19. If the point x is strongly comparable in limit with the point y, then it is comparable in limit with y. The inverse fact, generally speaking, does not take place.

Theorem 2.5.1. *Let a point y be Lagrange stable (stable in the sense of Lagrange). Then:*

(1) *if $(y; q_1, q_2)$ is a heteroclinic point and x is comparable in limit with y, then there exist points $p_1, p_2 \in P(X)$ such that $(x; p_1, p_2)$ is heteroclinic and, besides, the point p_1 (resp., p_2) is uniformly comparable with the point q_1 (resp., q_2);*

(2) *if $(y; q)$ is a homoclinic point and x is strongly comparable in limit with the point y, then there exists $p \in P(X)$ such that $(x; p)$ is a homoclinic point and the point p is uniformly comparable with q.*

Proof. (1) Let $\mathfrak{L}_y^{\infty} \subseteq \mathfrak{L}_x^{\infty}$ and $y \in W^u(q_1) \cap W^s(q_2)$. Then $\mathfrak{L}_y^{\pm\infty} \subseteq \mathfrak{L}_x^{\pm\infty}$ and, according to Corollary 2.8, there exist continuous mappings

$$h_1 : \alpha_y \longrightarrow \alpha_x, \qquad h_2 : \omega_y \longrightarrow \omega_x \tag{2.41}$$

such that $x \in W^s(p_1) \cap W^u(p_2)$, $h_1(\sigma(t, q)) = \pi(t, h_1(q))$ $(q \in \alpha_y, t \in \mathbb{T})$, and $h_2(\sigma(t, q)) = \pi(t, h_2(q))$ $(q \in \omega_y, t \in \mathbb{T})$, where $p_i = h_i(q_i)$ $(i = 1, 2)$.

In virtue of the compactness of $H(y)$, the mappings h_1 and h_2 are uniformly continuous and, consequently, the point p_i is uniformly comparable with q_i $(i = 1, 2)$.

(2) Let $\mathfrak{L}_y \subseteq \mathfrak{L}_x$ and $y \in W^u(q) \cap W^s(q)$. Then $\mathfrak{L}_y^{\pm\infty} \subseteq \mathfrak{L}_x^{\pm\infty}$ and, according to the first statement of the theorem, the point x is heteroclinic, that is, there exist $p_1, p_2 \in P(X)$ such that $x \in W^u(p_1) \cap W^s(p_2)$ and p_1, p_2 are uniformly comparable with q. Let us show that $p_1 = p_2$. For this aim we choose a sequence $\{t_n\} \in \mathfrak{L}_y$, for which $t_n \to \pm\infty$ as $n \to \pm\infty$ and $\sigma(t_n, q) \to q$ as $n \to \pm\infty$.

Note that $\{t_n\} \in \mathfrak{L}_x \cap \mathfrak{L}_{p_1} \cap \mathfrak{L}_{p_2}$, $xt_n \to p_1$ as $n \to -\infty$, and $xt_n \to p_2$ as $n \to +\infty$. Since $\{t_n\} \in \mathfrak{L}_x$, the sequence $\{xt_n\}$ is convergent and, consequently, $p_1 = p_2$. The theorem is proved. □

Definition 2.20. Let $(y; q_1, q_2)$ be a heteroclinic point. If the points q_1 and q_2 are stationary (resp., periodic, almost periodic, recurrent), the point y is called bilaterally asymptotically stationary (resp., bilaterally asymptotically periodic, bilaterally asymptotically almost periodic, bilaterally asymptotically recurrent).

Definition 2.21. If $(y; q)$ is a homoclinic point and q is stationary (resp., periodic, almost periodic, recurrent), then the point y is called stationary (resp., periodic, almost periodic, recurrent) homoclinic point.

From Theorem 2.5.1 follows the following statement.

Corollary 2.22. *Let $y \in Y$ be Lagrange stable. Then the following statements hold:*

(1) *if $(y; q_1, q_2)$ is bilaterally asymptotically stationary (resp., periodic, almost periodic, recurrent) and x is comparable in limit with y, then x is also bilaterally asymptotically stationary (resp., periodic, almost periodic, recurrent);*
(2) *if $(y; q)$ is a stationary (resp., periodic, almost periodic, recurrent) homoclinic point and x is strongly comparable in limit with y, then x is a stationary (resp., periodic, almost periodic, recurrent) homoclinic point.*

Let $h : X \to X$ be a homomorphism of the dynamical system $(X, \mathbb{T}, \pi)$ into $(Y, \mathbb{T}, \sigma)$. Let us consider an operator equation

$$h(x) = y, \tag{2.42}$$

where $y \in Y$. Along with (2.42) consider the family of limiting equations

$$h(x) = q \quad (q \in \Delta_y), \tag{2.43}$$

where $\Delta_y = \alpha_y \cup \omega_y$.

Theorem 2.5.2. *Let $y \in Y$ be Lagrange stable and $x \in X$ be a Lagrange stable solution of* (2.42)*. Then the following statements take place:*

(1) *if there are fulfilled the following conditions:*
 (a) *for any $q \in \omega_y$ (2.43) has at most one solution from ω_x;*
 (b) *for any $q \in \alpha_y$ (2.43) has at most one solution from α_x,*
 then the solution x is comparable in limit with y;
(2) *if* (2.43) *has at most one solution from Δ_x for any $q \in \Delta_y$, then the solution x is strongly comparable in limit with y.*

Proof. The first statement of the theorem, basically, follows from Theorem 2.3.1. The second one is proved using the same reasoning that in Theorem 2.3.1 and we do not give it here. □

Remark 2.23. Let a point y be Lagrange stable and x be a Lagrange solution of (2.42). Then the following statement hold:

(1) if the first condition of Theorem 2.5.2 is fulfilled and y is bilaterally asymptotically stationary (resp., periodic, almost periodic, recurrent), then x is also bilaterally asymptotically stationary (resp., periodic, almost periodic, recurrent);

(2) if the second condition of Theorem 2.5.2 is fulfilled and y is a asymptotically stationary (resp., periodic, almost periodic, recurrent) homoclinic point, then x is also a asymptotically stationary (resp., periodic, almost periodic, recurrent) homoclinic point.

Theorem 2.5.3. *Let x be a Lagrange stable solution of* (2.42) *and y is bilaterally asymptotically stationary (resp., periodic, almost periodic, recurrent), and for any $q \in \Delta_y$ solutions of* (2.43) *from Δ_x are separated in Δ_x. Then the solution x is bilaterally asymptotically stationary (resp., $m_0\tau$-periodic for a certain integer m_0, almost periodic, recurrent).*

Proof. The formulated statement follows from Theorems 2.3.2 and 2.4.3. □

2.6. Asymptotically Almost Periodic Systems with Convergence

Definition 2.24. Let $(X, \mathbb{T}, \pi)$ be a dynamical system on X. The system $(X, \mathbb{T}, \pi)$ is called [108, Chapter 1] point dissipative, if there exists a nonempty compact $K \subseteq X$ such that

$$\lim_{t \to +\infty} \rho(xt, K) = 0 \tag{2.44}$$

for all $x \in X$. In this case if (2.44) takes place uniformly with respect to x on compact subsets from X, then $(X, \mathbb{T}, \pi)$ is called compactly dissipative.

If the space X is locally compact, then from the point dissipativity of $(X, \mathbb{T}, \pi)$ it follows its compact dissipativity [109, Chapter I].

If the dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative, then there exists a maximal compact invariant set J, called the center of Levinson of $(X, \mathbb{T}, \pi)$, which is a orbitally stable global attractor [110] of the system $(X, \mathbb{T}, \pi)$ and $J := D(\Omega_X)$ [108, 111] where $\Omega_X = \overline{\cup\{\omega_x \mid x \in X\}}$ and $D(\Omega_X)$ is the positive prolongation [110] of Ω_X.

Remark 2.25. (1) Let $\mathbb{T}_+ \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_2$, $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two compactly dissipative dynamical systems and $h : X \to Y$ be a homomorphism of $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$. Then $J_Y \supseteq h(J_X)$, where J_X (resp., J_Y) is the center of Levinson of $(X, \mathbb{T}, \pi)$ (resp., $(Y, \mathbb{T}, \sigma)$).

(2) Let $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ be a nonautonomous dynamical system and the systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be compactly dissipative. Then $h : J_X \to J_Y$ is a homeomorphism, if h is one-to-one, that is, $X_y \cap J_X$ consists of exactly one point whatever would be $y \in J_Y$, where $X_y := \{x \mid x \in X, h(x) = y\}$.

Definition 2.26. A nonautonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is called convergent, if the following conditions are fulfilled:

(1) $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ are compactly dissipative;

(2) $J_X \cap X_y$ contains exactly one point, which we will denote by x_y (i.e., $J_X \cap X_y = \{x_y\}$), for any $y \in J_Y$.

Below, in this chapter and the following ones, we will suppose that $(Y, \mathbb{T}_2, \sigma)$ is compactly dissipative and J_Y is minimal (i.e., every trajectory from J_Y is dense in J_Y). It is so, if there exists an asymptotically almost periodic (resp., asymptotically recurrent) point $y_0 \in Y$ such that $Y = H^+(y_0) = \overline{\{y_0 t \mid t \in \mathbb{T}_+\}}$. Obviously, in this case $J_Y = \omega_{y_0}$.

Lemma 2.27. *Let $(X, \mathbb{T}_1, \pi)$ be compactly dissipative. Then the following conditions are equivalent:*

(1) *$\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is a convergent nonautonomous dynamical system;*

(2) (a) *$\lim_{t \to +\infty} \rho(x_1 t, x_2 t) = 0$ for every $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$;*

(b) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that from the inequality $\rho(x_1, x_2) < \delta$ ($h(x_1) = h(x_2)$ and $x_1, x_2 \in J_X$) follows that $\rho(x_1 t, x_2 t) < \varepsilon$ for all $t \in \mathbb{T}_+$.*

Proof. Let us show that (1) implies (2). First of all, let us establish that (a) is true. If we suppose that it does not take place, then there are $\varepsilon_0 > 0$, $y_0 \in Y$, $x_1, x_2 \in X_{y_0}$, and $t_k \to +\infty$ such that

$$\rho(x_1 t_k, x_2 t_k) \geq \varepsilon. \tag{2.45}$$

Without loss of generality we can suppose that the sequences $\{x_i t_k\}$ $(i = 1, 2)$, and $\{y_0 t_k\}$ are convergent. Assume $\overline{x}_i := \lim_{k \to +\infty} x_i t_k$ $(i = 1, 2)$ and $\overline{y} := \lim_{n \to +\infty} y_0 t_n$. Note that $\overline{y} \in J_Y$ and $\overline{x}_i \in J_X$ $(i = 1, 2)$. Besides, $h(\overline{x}_1) = \lim_{k \to +\infty} h(x_1) t_k = \lim_{k \to +\infty} h(x_2) t_k = h(\overline{x}_2) = \lim_{k \to +\infty} y_0 t_k = \overline{y}$ and, consequently, $\overline{x}_1, \overline{x}_2 \in J_X \cap X_{\overline{y}}$. In virtue of the convergence of $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ we have $\overline{x}_1 = \overline{x}_2$, and this contradicts to (2.45). The statement (a) is proved.

Now let us prove (b). Suppose the contrary. If (b) does not take place, then there exist $\varepsilon_0 > 0$, sequences $\delta_n \downarrow 0$, $\{x_k^i\}$ $(i = 1, 2)$ and $t_k \to +\infty$ such that $\rho(x_k^1, x_k^2) < \delta_k$ ($x_k^i \in J_X, h(x_k^1) = h(x_k^2)$) and

$$\rho(x_k^1 t_k, x_k^2 t_k) \geq \varepsilon_0. \tag{2.46}$$

By the compact dissipativity of $(X, \mathbb{T}_1, \pi)$ the sequences $\{x_k^i\}$ and $\{x_k^i t_k\}$ $(i = 1, 2)$ can be considered convergent. Assume $x_0 := \lim_{k \to +\infty} x_k^1 = \lim_{k \to +\infty} x_k^2$ and $x^i := \lim_{k \to +\infty} x_k^i t_k$ $(i = 1, 2)$. Note that $x_i \in J_X$ (see [111, 112]). In addition, $h(x^1) = \lim_{k \to +\infty} h(x_k^1) t_k = \lim_{k \to +\infty} h(x_k^2) t_k = h(x^2)$, that is, there exists $\overline{y} \in J_Y$ ($\overline{y} = h(x^1) = h(x^2)$) such that $x^1, x^2 \in J_X \cap X_{\overline{y}}$. Since $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is convergent, we have $x^1 = x^2$. This contradicts to (2.46).

Inversely. Let (2) be fulfilled. Let us show that (1) takes place. If we suppose that it is not so, then there exist $y_0 \in J_Y$ and $x_1, x_2 \in J_X \cap X_{y_0}$ $(x_1 \neq x_2)$. According to [113, Theorem 1], the points x_1 and x_2 are mutually recurrent and, consequently, the function $\varphi(t) := \pi(x_1 t, x_2 t)$ (for all $t \in \mathbb{T}$) is recurrent. On the other hand, under the conditions of the lemma, $\varphi(t) \to 0$ as $t \to +\infty$. Frow this it follows that $\varphi(t) \equiv 0$. The last contradicts to our assumption. The obtained contradiction completes the proof. □

Theorem 2.6.1. *A nonautonomous dynamical system $\langle (X,\mathbb{T}_1\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is convergent if and only if the following conditions hold:*

(1) *for any compact $K \subseteq X$ the set $\Sigma_K^+ := \{xt : x \in K, t \in \mathbb{T}_+\}$ is relatively compact;*
(2) *for any $\varepsilon > 0$ and compact subset $K \subseteq X$ there exists $\delta = \delta(\varepsilon, K) > 0$ such that $\rho(x_1,x_2) < \delta$ $(h(x_1) = h(x_2)$ and $x_1,x_2 \in K)$ implies $\rho(x_1t,x_2t) < \varepsilon$ for all $t \geq 0$;*
(3) $\lim_{t\to+\infty} \rho(x_1t,x_2t) = 0$ *for all $x_1,x_2 \in X$ $(h(x_1) = h(x_2))$.*

Proof. The necessity of the first condition is obvious. The necessity of the second and third statements follows from Lemma 2.27.

Inversely. Let conditions (1)–(3) of the theorem be fulfilled and $x_0 \in X$. According to condition (1), the set $\Sigma_{x_0}^+$ is relatively compact and, consequently, $\omega_{x_0} \neq \varnothing$ is compact and invariant. Since $h(\omega_{x_0}) \subseteq \Omega_Y \subseteq J_Y$ and J_Y is minimal, then $h(\omega_{x_0}) = J_Y$. Assume $\mathbf{M} := \omega_{x_0}$. Then $\mathbf{M}_y := \mathbf{M} \cap X_y$ $(y \in J_Y)$ is not empty. Let us show now that for any $x \in X$ the equality $\omega_x = \mathbf{M}$ takes place. Denote by $\mathbf{N} := \omega_x \cup \omega_{x_0}$. In the same way that in Lemma 2.27 we prove that $\mathbf{N}_y := \mathbf{N} \cap X_y$ consists of exactly one point for arbitrary $y \in J_Y$. Since $h(\omega_x) = h(\omega_{x_0}) = h(\mathbf{N}) = J_Y$, then $\omega_x \cap X_y = \omega_{x_0} \cap X_y = \mathbf{N} \cap X_y$ for all $y \in J_Y$ and, consequently, $\omega_x = \omega_{x_0}$. So, $\omega_x = \mathbf{M}$ for all $x \in X$ and, consequently, $(X,\mathbb{T},\pi)$ is point dissipative.

Now let $K \subseteq X$ be an arbitrary compact subset. According to condition (1), Σ_K^+ is relatively compact and, according to [112, 114], $\Omega(K) \neq \varnothing$ is compact, invariant and

$$\lim_{t\to+\infty} \sup \{\rho(xt,\Omega(K)) : x \in K\} = 0, \tag{2.47}$$

where

$$\Omega(K) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau \geq t} \pi^\tau K}. \tag{2.48}$$

In the same way that in Lemma 2.27 we show that $(\mathbf{N} \cup \Omega(K)) \cap X_y$ consists of exactly one point for arbitrary $y \in J_Y$. Since $h(\Omega((K)) = J_Y$, then $\Omega(K) = \mathbf{M}$. So, the system $(X,\mathbb{T}_1,\pi)$ is compactly dissipative and $\Omega(K) = \mathbf{M}$ for every compact $K \subseteq \mathbf{M}$, and, consequently, $J_X = \mathbf{M}$. As $\mathbf{M}_y$ consists of exactly one point for any $y \in J_Y$, then the nonautonomous system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is convergent. The theorem is proved. □

Corollary 2.28. *Let $(X,\mathbb{T}_1,\pi)$ be locally compact (i.e., for every $x \in X$ there exist $\delta_x > 0$ and $l_x > 0$ such that $\pi^t B(x,\delta_x)$ is locally compact for all $t \geq l_x$). A nonautonomous dynamical system $\langle (X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is convergent if and only if the following three conditions are fulfilled:*

(1) *for every $x \in X$ the set Σ_x^+ is relatively compact;*
(2) *for every $\varepsilon > 0$ and compact subset $K \subseteq X$ there exists $\delta = \delta(\varepsilon, K) > 0$ such that $\rho(x_1,x_2) < \delta$ $(h(x_1) = h(x_2)$ and $x_1,x_2 \in K)$ implies $\rho(x_1t,x_2t) < \varepsilon$ for all $t \geq 0$;*
(3) $\lim_{t\to+\infty} \rho(x_1t,x_2t) = 0$ *for all $x_1,x_2 \in X$ $(h(x_1) = h(x_2))$.*

Proof. The necessity of condition (1) is obvious and conditions (2) and (3) follow from Lemma 2.27. Concerning the sufficiency, it follows from Theorem 2.6.1. We should note that from conditions (1)–(3) follows the point dissipativity of $(X,\mathbb{T}_1,\pi)$ and in virtue of the local compactness of $(X,\mathbb{T}_1,\pi)$, according to [112], it is compactly dissipative. It means that condition (1) of Theorem 2.6.1 is fulfilled. □

Remark 2.29. If a space X is locally compact, then the system $(X,\mathbb{T}_1,\pi)$ is locally compact. Obviously, the inverse statement does not take place.

Theorem 2.6.2. *A nonautonomous dynamical system $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ is convergent if and only if the following conditions hold:*

(1) *Σ_K^+ is relatively compact for any compact subset K from X;*

(2) *every semitrajectory Σ_x^+ is asymptotically stable, that is,*

  (a) *for every $\varepsilon>0$ and $\overline{x}\in X$ there exists $\delta=\delta(\varepsilon,\overline{x})>0$ such that $\rho(x,\overline{x})<\delta$ $(h(x)=h(\overline{x}))$ implies $\rho(xt,\overline{x}t)<\varepsilon$ for all $t\in\mathbb{T}_+$;*

  (b) *there exists $\gamma(\overline{x})>0$ such that $\rho(x,\overline{x})<\gamma(\overline{x})$ $(h(x)=h(\overline{x}))$ implies $\lim_{t\to+\infty}\rho(xt,\overline{x}t)=0$.*

Proof. Let $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$ be convergent. Condition (1) is obvious. Let us show that every semitrajectory is asymptotically stable. If we suppose that it is not so, then there exist $x_0\in X,\varepsilon_0>0,x_k\to x_0$ $(h(x_k)=h(x_0))$, and $t_k\to+\infty$ such that

$$\rho(x_kt_k,x_0t_k)\geq\varepsilon_0. \tag{2.49}$$

Since $(X,\mathbb{T}_1,\pi)$ is compactly dissipative, then the sequences $\{x_kt_k\}$ and $\{x_0t_k\}$ can be considered convergent. Put $\overline{x}:=\lim_{k\to+\infty}x_kt_k$ and $\overline{\overline{x}}:=\lim_{k\to+\infty}x_0t_k$. From (2.49) it follows that $\overline{x}\neq\overline{\overline{x}}$. Note, that $\overline{x},\overline{\overline{x}}\in D(\Omega_X)=J_X$. Without loss of generality $\{y_0t_k\}$ can be considered convergent. Assume $\overline{y}:=\lim_{k\to+\infty}y_0t_k$. Then $h(\overline{x})=\lim_{k\to+\infty}h(x_k)t_k=\lim_{k\to+\infty}y_0t_k=\overline{y}$ and $h(\overline{\overline{x}})=\lim_{k\to+\infty}h(x_0)t_k=\lim_{k\to+\infty}y_0t_k=\overline{y}$. From this follows that $\overline{x},\overline{\overline{x}}\in J_X\cap X_{\overline{y}}$ $(\overline{y}\in J_Y)$. On the other hand, by the convergence of $\langle(X,\mathbb{T}_1,\pi),(Y,\mathbb{T}_2,\sigma),h\rangle$, the set $J_X\cap X_{\overline{y}}$ contains at most one point. Consequently, $\overline{x}=\overline{\overline{x}}$. The last contradicts to (2.49) and so the asymptotical stability of every semitrajectory of Σ_x^+ is proved. Condition (b) follows from Lemma 2.27.

Inversely. First of all, let us show that if $\overline{x}\in X_{\overline{y}}$ $(\overline{y}:=h(\overline{x}))$, then

$$\lim_{t\to+\infty}\rho(xt,\overline{x}t)=0 \tag{2.50}$$

for all $x\in X_{\overline{y}}$. Suppose that it is not so. Denote by $G_{\overline{y}}$ the set of all the points $x\in X_{\overline{y}}$, for which (2.50) takes place. In virtue of our assumption $G_{\overline{y}}\neq X_{\overline{y}}$. Note that under the conditions of Theorem 2.6.2, $G_{\overline{y}}$ is open in $X_{\overline{y}}$. Assume $\Gamma_{\overline{y}}:=\partial G_{\overline{y}}$ and let $\overline{\overline{x}}\in\Gamma_{\overline{y}}$. Then $B(\overline{\overline{x}},\gamma(\overline{\overline{x}}))\cap G_{\overline{y}}\neq\varnothing$ and $B(\overline{\overline{x}},\gamma(\overline{\overline{x}}))\cap(X_{\overline{y}}\setminus G_{\overline{y}})\neq\varnothing$. It is easy to see that these relations cannot be held simultaneously and, consequently, $\Gamma_{\overline{y}}=\varnothing$ for every $\overline{y}\in Y$.

Let us show now that for arbitrary compact $K\subseteq X$ and $\varepsilon>0$ there exists $\delta=\delta(\varepsilon,K)>0$ such that $\rho(x_1,x_2)<\delta$ $(h(x_1)=h(x_2)$ and $x_1,x_2\in K)$ implies $\rho(x_1t,x_2t)<\varepsilon$ for all $t\in\mathbb{T}_+$. Suppose the contrary. Then there exists a compact subset $K_0\subseteq X$, $\varepsilon_0>0$, $\delta_n\downarrow 0$, $\{x_k^i\}\subseteq K_0$ $(i=1,2,\ h(x_k^1)=h(x_k^2))$, and $t_k\to+\infty$ such that

$$\rho(x_k^1,x_k^2)<\delta_k,\qquad \rho(x_k^1t_k,x_k^2t_k)\geq\varepsilon_0. \tag{2.51}$$

By the compactness of K_0 we consider that the sequences $\{x_k^i\}$ $(i=1,2)$ are convergent. Put $\overline{x}:=\lim_{k\to+\infty}x_k^1=\lim_{k\to+\infty}x_k^2\in K_0$. Since the semitrajectory $\Sigma_{\overline{x}}^+$ is asymptotically stable, then for $\varepsilon_0/3$ and $\overline{x}$ there exists $\delta(\varepsilon_0/3,\overline{x})>0$ such that $\rho(x,\overline{x})<\delta(\varepsilon_0/3,\overline{x})$ $(h(x)=h(\overline{x}))$ implies $\rho(xt,\overline{x}t)<\varepsilon/3$ for all $t\in\mathbb{T}_+$. Since $x_k^i\to\overline{x}$ $(i=1,2)$ as $k\to+\infty$, there exists

$\bar{k} \in \mathbb{N}$ such that $\rho(x_k^i, \bar{x}) < \delta(\varepsilon_0/3, \bar{x})$ for all $k \geq \bar{k}$ and, consequently, $\rho(x_k^i t, \bar{x}t) < \varepsilon_0/3$ for all $t \in \mathbb{T}_+$. From the last inequality we obtain

$$\rho(x_k^1 t, x_k^2 t) \leq \frac{2\varepsilon_0}{3} < \varepsilon_0 \tag{2.52}$$

for all $t \in \mathbb{T}_+$ and $k \geq \bar{k}$. Since (2.51) contradicts to (2.52), the necessary statement is proved. Now, to finish the proof of Theorem 2.6.2 it is sufficient to refer to Theorem 2.6.1. □

The following theorem takes place.

Theorem 2.6.3. *Let a point $y_0 \in Y$ be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) so that $Y = H^+(y_0)$ and the nonautonomous dynamical system $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ be convergent. Then the following statements take place:*

(1) *the Levinson center J_X of the dynamical system $(X, \mathbb{T}_1, \pi)$ is homeomorphic to ω_{y_0} and, consequently, it is a minimal set consisting of the stationary motion (resp., τ-periodic motions, almost periodic motions, recurrent motions);*
(2) *any point $x \in X$ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent), and $\omega_x = J_X$ for all $x \in X$ and, consequently, $W^s(J_X) = X$;*
(3) *for every $\varepsilon > 0$ and compact subset $K \subseteq X$ there exists $\delta = \delta(\varepsilon, K) > 0$ such that $\rho(x_1, x_2) < \delta$ implies $\rho(x_1 t, x_2 t) < \varepsilon$ for all $t \in \mathbb{T}_+$ and $x_1, x_2 \in K$, for which $h(x_1) = h(x_2)$.*

Proof. Under the condition of Theorem 2.6.3 the dynamical system $(Y, \mathbb{T}, \sigma)$ is compactly dissipative and $J_X = \omega_{y_0}$. Thus, if $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is convergent, then J_X and $J_Y = \omega_{y_0}$ are homeomorphic and, consequently, J_X is a minimal set consisting of the stationary motion (resp., τ-periodic motions, almost periodic motions, recurrent motions).

Since $\omega_x \subseteq J_X$ for every $x \in X$, then in virtue of the minimality of J_X we have $\omega_x = J_X$. As $J_X \cap X_y$ consists exactly of one point for every $y \in J_Y = \omega_{y_0}$ and $\omega_{h(x)} = \omega_{y_0}$, then according to Theorem 2.3.1 the point x is comparable in limit with $y = h(x)$. Since every point $y \in H^+(y_0) = Y$ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent), then from Corollary 2.9 it follows that the point x possesses the same property.

The third statement of the theorem it follows from Theorem 2.6.1. □

2.7. Some Tests of Convergence

Definition 2.30. A set $\mathbf{M} \subseteq X$ is called uniformly stable with respect to the homomorphism $h : X \to Y$ (un. st. h), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(x_1, x_2) < \delta$ implies $\rho(x_1 t, x_2 t) < \varepsilon$ for all $t \in \mathbb{T}_+$ and $x_1, x_2 \in \mathbf{M}$, for which $h(x_1) = h(x_2)$. If X un. st. h, then the dynamical system $(X, \mathbb{T}_1, \pi)$ is called uniformly stable with respect to the homomorphism h.

Lemma 2.31. *Let a homomorphism $h : X \to Y$ satisfy the following conditions:*

(1) *there exists a continuous section $\gamma : Y \to X$, that is, there exists a continuous mapping $\gamma : Y \to X$ for which $h \circ \gamma = Id_Y$;*

(2) $\lim_{t\to+\infty} \rho(x_1t, x_2t) = 0$ *for all $x_1, x_2 \in X$ $(h(x_1) = h(x_2))$.*

Then the following statements hold:

(1) *if $(Y, \mathbb{T}_2, \sigma)$ is point dissipative, then $(X, \mathbb{T}_1, \pi)$ is point dissipative and in this case Ω_X and Ω_Y are homeomorphic;*

(2) *if $(Y, \mathbb{T}_2, \sigma)$ is compactly dissipative and every compact subset $K \subseteq X$ is uniformly stable with respect to h, then $(X, \mathbb{T}_1, \pi)$ is compactly dissipative, and J_X and J_Y are homeomorphic and, consequently, $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h\rangle$ is convergent.*

Proof. Let $(Y, \mathbb{T}_1, \sigma)$ be point dissipative. Then $\Omega_Y = \overline{\cup\{\omega_y \mid y \in Y\}}$ is a nonempty compact invariant set and, consequently, $\mathbf{M} := \gamma(\Omega_Y) \subseteq \Omega_X$ also is nonempty, compact and invariant. For $x \in X$ and $y := h(x)$ we have $\lim_{t\to+\infty} \rho(xt, \gamma(y)t) = 0$. Hence, Σ_x^+ is a relatively compact set. Besides, $\omega_X \subseteq \omega_{\gamma(Y)} \subseteq \gamma(\Omega_Y) = \mathbf{M}$. From this it follows that $\Omega_X \subseteq \mathbf{M}$. So, $(X, \mathbb{T}_1, \pi)$ is point dissipative and $\gamma(\Omega_Y) = \Omega_X$. Since $\gamma : \Omega_Y \to \Omega_X$ separates points and Ω_Y is compact, then Ω_Y and Ω_X are homeomorphic.

Let $(Y, \mathbb{T}_2, \sigma)$ be compactly dissipative. Then, according to the said above, $(X, \mathbb{T}, \pi)$ is point dissipative. Let us assume that $\mathbf{M} := \gamma(J_Y)$ and we will show that $\mathbf{M}$ is orbitally stable. Suppose that it is not so. The there exist $\varepsilon_0 > 0, x_k \to x_0 \in \mathbf{M}$, and $t_k \to +\infty$ such that

$$\rho(x_kt_k, \mathbf{M}) \geq \varepsilon_0. \tag{2.53}$$

Note that $h(x_k) = y_k \to y_0 := h(x_0) \in h(\mathbf{M}) = h \circ \gamma(J_Y) = J_Y$ and, in virtue of the compact dissipativity of $(Y, \mathbb{T}_2, \sigma)$ the sequence $\{y_kt_k\}$ can be considered convergent. Put $y := \lim_{k\to+\infty} y_kt_k$. It is clear that $y \in J_Y$ and $\gamma(y) = \lim_{k\to+\infty} \gamma(h(x_k))t_k$. Since $\gamma : J_Y \to \gamma(J_Y) = \mathbf{M}$ separates points and $h\circ\gamma = Id_Y$, then $\gamma : J_Y \to \mathbf{M}$ is a homomorphism and $\gamma \circ h(x) = x$ for all $x \in \mathbf{M}$ and, consequently, $\gamma(h(x_k)) \to \gamma(h(x_0)) = x_0 \in \mathbf{M}$. From the last relation it follows that

$$\lim_{k\to+\infty} \rho(x_k, \gamma \circ h(x_k)) = 0. \tag{2.54}$$

Let $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ be the numbers from the uniform stability of the compact set $K = \overline{\{x_k\}} \cup \overline{\gamma \circ h\{x_k\}}$ with respect to the homomorphism h. From (2.54) it follows that for k large enough there takes place $\rho(x_k, \gamma \circ h(x_k)) < \delta$ and, consequently, $\rho(x_kt, (\gamma \circ h)(x_k)t) < \varepsilon$ for all $t \in \mathbb{T}_+$. In particular,

$$\rho(x_kt_k, \gamma \circ h(x_k)t_k) < \varepsilon \tag{2.55}$$

for k large enough. Since ε is arbitrary, from (2.55) it follows that $\lim_{k\to+\infty} x_kt_k = \lim_{k\to+\infty} \gamma \circ h(x_k)t_k = \gamma(y) \in \mathbf{M}$. The last contradicts to (2.53). The obtained contradiction shows that $\mathbf{M}$ is orbitally stable. So, $(X, \mathbb{T}, \pi)$ is point dissipative, $\Omega_X \subseteq \mathbf{M}$, $\mathbf{M}$ is nonempty, compact, invariant, and orbitally stable. According to [111, Lemma 7], $(X, \mathbb{T}_1, \pi)$ is compactly dissipative and $J_X \subseteq \mathbf{M}$. To finish the proof of the lemma it is sufficient to refer to Lemma 2.27 and Remark 2.25. □

Let $(Y,\mathbb{S},\sigma)$ be a group dynamical system, $(X,\mathbb{S}_+,\pi)$ be a semigroup dynamical system, and $h : X \to Y$ be a homomorphism of $(X,\mathbb{S}_+,\pi)$ onto $(Y,\mathbb{S},\sigma)$. Let us consider the nonautonomous dynamical system $\langle(X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ and denote by $\Gamma(Y,X)$ the set of all continuous sections of the homomorphism h. The equality

$$d(\gamma_1,\gamma_2) = \sup_{y\in Y}\rho(\gamma_1(y),\gamma_2(y)) \tag{2.56}$$

defines a full metric on $\Gamma(Y,X)$.

Assume $X\dot{\times}X := \{(x_1,x_2) \mid x_1,x_2 \in X,\ h(x_1) = h(x_2)\}$ and let $V : X\dot{\times}X \to \mathbb{R}_+$ be a mapping satisfying the following conditions:

(a) $a(\rho(x_1,x_2)) \le V(x_1,x_2) \le b(\rho(x_1,x_2))$ for all $(x_1,x_2) \in X\dot{\times}X$, where $a,b \in \mathcal{K}$ and $\operatorname{Im} a = \operatorname{Im} b$ ($\mathcal{K} := \{a \mid a : \mathbb{R}_+ \to \mathbb{R}_+, a$ is continuous, strictly monotone increasing, and $a(0) = 0\}$);
(b) $V(x_1,x_2) = V(x_2,x_1)$ for all $(x_1,x_2) \in X\dot{\times}X$;
(c) $V(x_1,x_2) \le V(x_1,x_3)+X(x_3,x_2)$ for all $x_1,x_2,x_3 \in X$ such that $h(x_1) = h(x_2) = h(x_3)$;
(d) there exist $N > 0$ and $\gamma > 0$ such that

$$V(x_1t,x_2t) \le Ne^{-\gamma t}V(x_1,x_2) \quad (\forall\,(x_1,x_2) \in X\dot{\times}X, t \in \mathbb{S}_+). \tag{2.57}$$

From conditions (a)–(c) it follows that on every fiber $X_y := h^{-1}(y)$ the mapping V defines a metric that is topologically equivalent to ρ. The following lemma takes place.

Lemma 2.32 (see [115]). *Let $V : X\dot{\times}X \to \mathbb{R}_+$ satisfy the conditions (a)–(c). Then on $\Gamma(Y,X)$ the equality*

$$p(\gamma_1,\gamma_2) = \sup_{y\in Y} V(\gamma_1(y),\gamma_2(y)) \tag{2.58}$$

defines a complete metric, topologically equivalent to (2.56).

Lemma 2.33. *Let $\langle(X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ be a nonautonomous dynamical system satisfying the following conditions:*

(1) $\Gamma(Y,X) \neq \emptyset$;
(2) *there exists a function $V : X\dot{\times}X \to \mathbb{R}_+$ satisfying the conditions (a)–(c)*

and

(3) $V(x_1t,x_2t) \le Ne^{-\gamma t}V(x_1,x_2)$ $(\forall\,(x_1,x_2) \in X\dot{\times}X,\ t \in \mathbb{S}_+)$,

where $N,\gamma > 0$.

Then there exists a unique invariant continuous section of h, that is, there exists $\gamma \in \Gamma(Y,X)$, such that $\pi^t \circ \gamma = \gamma \circ \sigma^t$ for all $t \in \mathbb{T}_+$.

Proof. Let us denote by $S^t : \Gamma(Y,X) \to \Gamma(Y,X)$ the mapping defined by the equality $(S^t\gamma)(y) := \pi^t\gamma(\sigma^{-t}y)$ for all $t \in \mathbb{S}_+, \gamma \in \Gamma(Y,X)$ and $y \in Y$. It is easy to check that $\{S^t\}_{t\geq 0}$ is a commutative semigroup with respect to the composition. Note that

$$p(S^t\gamma_1, S^t\gamma_2) = \max_{y\in Y} V(\pi^t\gamma_1(\sigma^{-t}y), \pi^t\gamma_2(\sigma^{-t}y)) \\ \leq Ne^{-\gamma t}\max_{y\in Y} V(\gamma_1(\sigma^{-t}y), \gamma_2(\sigma^{-t}y)) \leq Ne^{-\gamma t}p(\gamma_1,\gamma_2). \tag{2.59}$$

From inequality (2.59) it follows that the mappings S^t are contractions for $t \in \mathbb{S}_+$ large enough. From this, in virtue of the commutativity of $\{S^t\}_{t\geq 0}$, it follows that there exists a common fixed point γ of the semigroup $\{S^t\}_{t\geq 0}$ which is an invariant section of h, that is, $\pi^t \circ \gamma = \gamma \circ \sigma^t$ for all $t \in \mathbb{S}_+$. □

Theorem 2.7.1. *Let a nonautonomous dynamical system $\langle(X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ satisfy the conditions:*

(1) $\Gamma(Y,X) \neq \emptyset$.
(2) *there exists a function $V : X\dot{\times}X \to \mathbb{R}_+$ satisfying the conditions (a)–(d).*

Then $(X,\mathbb{S}_+,\pi)$ is compactly dissipative, and J_X and J_Y are homeomorphic. Consequently, $\langle(X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is convergent.

Proof. Note, that for $x_1,x_2 \in X$ $(h(x_1) = h(x_2))$,

$$a(\rho(x_1t,x_2t)) \leq V(x_1t,x_2t) \leq Ne^{-\gamma t}V(x_1,x_2) \leq Ne^{-\gamma t}b(\rho(x_1,x_2)). \tag{2.60}$$

Hence, $\lim_{t\to+\infty} a(\rho(x_1t,x_2t)) = 0$ and $\lim_{t\to+\infty}\rho(x_1t,x_2t) = 0$. Let $\varepsilon > 0$ and $\delta(\varepsilon) = b^{-1}(N^{-1}a(\varepsilon))$. Since $\rho(x_1,x_2) < \delta(\varepsilon)$ $(h(x_1) = h(x_2))$ implies $\rho(x_1t,x_2t) < \varepsilon$ for all $t \in \mathbb{S}_+$, the system $(X,\mathbb{S}_+,\pi)$ is uniformly stable with respect to h. Now to complete the proof of the theorem it is sufficient to refer to Lemmas 2.31 and 2.33. □

Remark 2.34. Lemma 2.32 and Theorem 2.7.1 take place also when Y is not compact. In this case we will denote by $\Gamma(Y,X)$ the set of all continuous bounded sections.

Corollary 2.35. *Let $y_0 \in Y$ be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) and $Y = H(y_0)$. If for the dynamical system $\langle(X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ the conditions of Theorem 2.7.1 are fulfilled, then it is convergent and, besides,*

(1) *the Levinson center J_X of the dynamical system $(X,\mathbb{S}_+,\pi)$ is homeomorphic to ω_{y_0} and consists of the stationary motion (resp., τ-periodic motions, almost periodic motions, recurrent motions).*
(2) *any point $x \in X$ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).*
(3) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(x_1,x_2) < \delta$ implies $\rho(x_1t,x_2t) < \varepsilon$ for all $t \in \mathbb{S}_+$ and $x_2,x_2 \in X$ for which $h(x_1) = h(x_2)$.*

Proof. The formulated statement it follows from Theorems 2.7.1, 2.6.2, and Remark 2.34. □

In conclusion, note that convergent dynamical systems are in a way the simplest dissipative dynamical systems. If a nonautonomous system $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle$ is convergent and J_X (resp., J_Y) is the Levinson center of $(X, \mathbb{S}_+, \pi)$ (resp., $(Y, \mathbb{S}, \sigma)$), then J_X and J_Y are homeomorphic. From this we see that on the one hand, the Levinson center of a convergent system can be described fully enough and, on the other hand, it can be notably complicated.

3 Asymptotically Almost Periodic Solutions of Ordinary Differential Equations

3.1. Some Nonautonomous Dynamical Systems

Example 3.1. Let E^n be an n-dimensional real or complex Euclidian space with the norm $|\cdot|$. Consider a differential equation

$$\frac{du}{dt} = f(t,u), \tag{3.1}$$

where $f \in C(\mathbb{R} \times E^n, E^n)$. Along with (3.1) let us consider also its H-class [92, 93, 98, 100, 107, 116]

$$\frac{dv}{dt} = g(t,v), \tag{3.2}$$

where $g \in H(f) = \overline{\{f^{(\tau)} : \tau \in \mathbb{R}\}}$ and $f^{(\tau)}(t,u) := f(t+\tau,u)$. In this example we suppose that the function f is regular, that is, for every (3.2) the conditions of existence, uniqueness, and nonlocal extensibility of its solutions on $\mathbb{R}_+$ are held. Denote by $\varphi(t,v,g)$ a solution of (3.2) passing through the point $v \in E^n$ as $t = 0$. Then the mapping $\varphi : \mathbb{R}_+ \times E^n \times H(f) \to E^n$ is well defined and the following conditions are fulfilled (see, i.e., [88, 89, 93]):

(1) $\varphi(0,v,g) = v$ (for all $v \in E^n$ and $g \in H(f)$;
(2) $\varphi(t,\varphi(\tau,v,g),g_\tau) = \varphi(t+\tau,v,g)$ (for all $v \in E^n, g \in H(f)$ and $t,\tau \in \mathbb{R}_+$);
(3) $\varphi : \mathbb{R}_+ \times E^n \times H(f) \to E^n$ is continuous.

Let us put $Y := H(f)$ and by $(Y,\mathbb{R},\sigma)$ denote a dynamical system of shifts on Y induced by the dynamical system of shifts $(C(\mathbb{R} \times E^n, E^n),\mathbb{R},\sigma)$. Put $X := E^n \times Y$ and define a mapping $\pi : X \times \mathbb{R}_+ \to X$ as follows: $\pi((v,g),\tau) = (\varphi(\tau,v,g),g^\tau)$ (i.e., $\pi := (\varphi,\sigma)$). Then it is easy to verify that $(X,\mathbb{R}_+,\pi)$ is a dynamical system of shifts on X and $h = pr_2 : X \to Y$ is a homomorphism of $(X,\mathbb{R}_+,\pi)$ onto $(Y,\mathbb{R},\sigma)$ and, consequently, $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ is a nonautonomous dynamical system generated by (3.1).

Remark 3.2. Also we will consider the case $Y = H^+(f) = \overline{\{f^{(\tau)} | \tau \in \mathbb{R}_+\}}$, a semi-group dynamical system $(H^+(f), \mathbb{R}_+, \sigma)$, and a semigroup nonautonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(H^+(f),\mathbb{R}_+,\sigma),h\rangle$.

Example 3.3. Let us consider differential equation (3.1) with the right-hand side $f \in C(\mathbb{R} \times W, E^n)$, where W is some open set from E^n. Denote $Y := C(\mathbb{R} \times W, E^n)$ and by $(Y, \mathbb{R}, \sigma)$ denote a dynamical system of shifts on Y. By X denote the set of all pairs (φ, f) from $C(\mathbb{R}_+, W) \times C(\mathbb{R} \times W, E^n)$ such that φ is a solution of (3.1). Obviously, X is invariant with respect to shifts in the dynamical system of shifts $(C(\mathbb{R}_+, E^n) \times C(\mathbb{R} \times W, E^n), \mathbb{R}_+, \pi)$, where $\pi((\varphi, f), \tau) = (\varphi^{(\tau)}, f^{(\tau)})$. From general properties of differential equations it follows the closeness of X in $C(\mathbb{R}_+, E^n) \times C(\mathbb{R} \times W, E^n)$ and, consequently, on X there is induced a dynamical system of shifts $(X, \mathbb{R}_+, \pi)$. The mapping $h := pr_2 : X \to Y$ is a homomorphism of the dynamical system $(X, \mathbb{R}_+, \pi)$ on $(Y, \mathbb{R}, \sigma)$. Hence $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a nonautonomous dynamical system.

Example 3.4. Let $\varphi \in C(\mathbb{R}_+, E^n)$ be a solution of (3.1), $Q := \overline{\varphi(\mathbb{R}_+)}$, $f_Q := f|_{\mathbb{R} \times Q}$ and $H(f_Q) := \overline{\sigma(f_Q, \mathbb{R})}$, where $\sigma(\cdot, f_Q)$ is a motion generated by f_Q in the dynamical system of shifts $(C(\mathbb{R} \times Q, E^n), \mathbb{R}, \sigma)$. Assume $Y := H(f_Q)$ and $X := H^+(\varphi, f_Q)$, where $H^+(\varphi, f_Q)$ is the closure of the positive semitrajectory of (φ, f_Q) in the product dynamical systems $(C(\mathbb{R}_+, E^n), \mathbb{R}_+, \sigma) \times (C(\mathbb{R} \times Q, E^n), \mathbb{R}, \sigma)$. Then the mapping $h := pr_2 : X \to Y$ is a homomorphism of $(X, \mathbb{R}_+, \pi)$ onto $(Y, \mathbb{R}, \sigma)$, where $(Y, \mathbb{R}, \sigma)$ (resp., $(X, \mathbb{R}_+, \pi)$) is a dynamical system on Y (resp., X) induced by the dynamical system $(C(\mathbb{R} \times Q, E^n, \mathbb{R}, \sigma)$ (resp., $(C(\mathbb{R}_+, E^n), \mathbb{R}_+, \sigma) \times (C(\mathbb{R} \times Q, E^n), \mathbb{R}, \sigma)$). So, $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a nonautonomous dynamical system.

Example 3.5. Let us denote by $L^p_{\text{loc}}(\mathbb{R} \times W, E^n)$ the space of all the functions $f : \mathbb{R} \times W \to E^n$ that satisfy the following two conditions (the conditions of Carathéodory):

(a) for every fixed $t \in \mathbb{R}$ the function f is continuous with respect to $x \in W$;

(b) for every fixed compact $Q \subseteq W$ there exists a positive function $m_Q \in L^p_{\text{loc}}(\mathbb{R}, \mathbb{R})$ such that

$$|f(t, x)| \leq m_Q(t)$$

for all $t \in \mathbb{R}$ and $x \in Q$.

Let us define with the help of family of seminorms a topology in the space $L^p_{\text{loc}}(\mathbb{R} \times W, E^n)$. The family of seminorms is defined as follows. Let $l > 0$ and Q be an arbitrary compact subset from W. Assume that

$$\|f\|^p_{[-l,l] \times Q} := \int_{|t| \leq l} \max_{x \in Q} |f(t, x)|^p dt. \tag{3.3}$$

Define a mapping $\sigma : L^p_{\text{loc}}(\mathbb{R} \times W, E^n) \times \mathbb{R} \to L^p_{\text{loc}}(\mathbb{R} \times W, E^n)$ by the equality $\sigma(\tau, f) := f^{(\tau)}$. We can show (see, e.g., [117]) that the triple $(L^p_{\text{loc}}(\mathbb{R} \times W, E^n), \mathbb{R}, \sigma)$ is a dynamical system.

The following lemma takes place.

Lemma 3.6. *Let Q be a compact subset from E^n, $S_1, S_2, \ldots, S_m, \ldots$ is an increasing sequence of intervals in $\mathbb{R}$, for which $\bigcup_{m=1}^{\infty} S_m = \mathbb{R}$ and $\varphi_m \in C(\mathbb{R}, Q)$ $(m \in \mathbb{N})$. Suppose that the following conditions are fulfilled:*

(1) *for every $m \in \mathbb{N}$ in $L^p_{\text{loc}}(S_m \times E^n, E^n)$ there exists a function f_m such that the restriction of the function φ_m on S_m is a solution of the differential equation*

$$\frac{dx}{dt} = f_m(t, x), \tag{3.4}$$

(2) *for every segment* $S \subseteq \mathbb{R}$ *and every* ε *there exists* $n_0 \in \mathbb{N}$ *such that* $S \subseteq S_m$ *and*

$$\int_{(S)} \max_{x \in Q} |f_m(t,x) - f(t,x)|\,dt < \varepsilon \tag{3.5}$$

for all $m \geq n_0$.

Then the following statements take place:

(1) *the set of functions* $\Phi = \{\varphi_n : n \in \mathbb{N}\}$ *is relatively compact in* $C(\mathbb{R}, Q)$*;*
(2) *the limit* ϕ *of every convergent subsequence of the sequence* $\{\varphi_n\}$ *is a Q-compact solution of* (3.1) *(i.e.,* $\varphi(\mathbb{R}) \subseteq Q$*) defined on the whole axis* $\mathbb{R}$*;*
(3) *if a real number* $t_0 \in \mathbb{R}$ *is such that the sequence* $\{\varphi_n(t_0)\} \subseteq E^n$ *converges to some point* $x_0 \in Q$ *and* φ *is the unique not extensible solution of* (3.1) *satisfying the initial condition* $\varphi(t_0) = x_0$*, that is, then* φ *is a Q-compact solution of* (3.1) *defined on the whole axis* $\mathbb{R}$ *and the sequence* $\{\varphi_n\}$ *converges to* φ *in the space* $C(\mathbb{R}, Q)$.

Proof. The formulated lemma is a generalization of [92, Lemma 3.1.5] and is proved in the same way. So, we omit its proof. □

Assume that $Y := L^p_{\text{loc}}(\mathbb{R} \times W, E^n)$ and by $(Y, \mathbb{R}, \sigma)$ denote a dynamical system of shifts on Y. By X denote the set of all pairs $(\varphi, f) \in C(\mathbb{R}_+, E^n) \times L^p_{\text{loc}}(\mathbb{R} \times W, E^n)$ such that φ is a solution of (3.1). From general properties of differential equations follows that X is closed and invariant in $C(\mathbb{R}_+, E^n) \times L^p_{\text{loc}}(\mathbb{R} \times W, E^n)$ and, consequently, on X there is induced a dynamical system of shifts $(X, \mathbb{R}_+, \pi)$. The mapping $h = pr_2 : X \to Y$ is a homomorphism of $(X, \mathbb{R}_+, \pi)$ onto $(Y, \mathbb{R}, \sigma)$ and, consequently, the triple $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a nonautonomous dynamical system.

Example 3.7. Let us denote by $CH(\mathbb{R} \times \mathbb{C}^n, \mathbb{C}^n)$ the set of all continuous with respect to $t \in \mathbb{R}$ and holomorphic with respect to $z \in \mathbb{C}^n$ functions $f : \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$, which is endowed with the topology of uniform convergence on compact subsets from $\mathbb{R} \times \mathbb{C}^n$. Consider (3.1) with the right-hand side $f \in CH(\mathbb{R} \times \mathbb{C}^n, \mathbb{C}^n)$ and its H-class. Denote by $\varphi(t, z, g)$ a solution of (3.2) passing through the point z as $t = 0$ and defined on $\mathbb{R}_+$. Note that the mapping $\varphi : \mathbb{R}_+ \times \mathbb{C}^n \times H(f) \to \mathbb{C}^n$ satisfies conditions (1)–(3) from Example 3.1 and, besides, for every $t \in \mathbb{R}_+$ and $g \in H(f)$ the mapping $U(t, g) = \varphi(t, \cdot, g) : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic [118]. Put $Y := H(f)$ and by $(Y, \mathbb{R}, \sigma)$ denote a dynamical system of shifts on Y. Let $X = \mathbb{C}^n \times Y$ and $(X, \mathbb{R}_+, \pi)$ be a dynamical system on X, where $\pi := (\varphi, \sigma)$. At last, if $h = pr_2 : X \to Y$, then $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a nonautonomous dynamical system. From the above mentioned fact that the mappings $U(t, g) : \mathbb{C}^n \to \mathbb{C}^n$ are holomorphic it follows that for any $y \in Y$ and $t \in \mathbb{R}^+$ the mapping $\pi^t : X_y \to X_{\sigma(t,y)}$ $(X_y := h^{-1}(y))$ is holomorphic.

3.2. Compatible in Limit Solutions

Let us consider the problem of dependence of the property of recurrence in limit of solutions for differential equations on the according property of the right-hand sides of equations. Namely, we will study the problem of asymptotically periodicity (resp., asymptotically almost periodicity, asymptotically recurrence) of solutions, if the right-hand side of equation possesses the same property.

It is well known [92] that, if the right-hand side of a differential equation is Poisson stable with respect to the time (resp., periodic, almost periodic, recurrent) function, then under certain conditions among bounded solutions of that equation there exists a solution which is compatible with respect to recurrence with the right-hand side.

So, we can see a notably common and deep dependence, in virtue of which the character of the recurrence of solutions of differential equations is compatible with the recurrence of the right-hand side of equations (see, e.g., [92, 93, 100]).

The given below results show that an analogous dependence of the recurrence in limit of the right-hand side of differential equations also takes place when the right-hand side is asymptotically Poisson stable.

Definition 3.8. Let $f \in C(\mathbb{R} \times W, E^n)$ and Q be a compact subset from W. One will say that the function f is asymptotically stationary (resp., τ-periodic, almost periodic, recurrent, stable in the sense of Poisson) with respect to the variable $t \in \mathbb{R}$ uniformly with respect to $x \in Q$, if the motion $\sigma(\cdot, f_Q)$ generated by the function $f_Q := f|_{\mathbb{R}\times Q}$ in the dynamical system of shifts $(C(\mathbb{R}\times Q, E^n), \mathbb{R}, \sigma)$ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent, asymptotically Poisson stable).

Remark 3.9. Function $f \in C(\mathbb{R} \times W, E^n)$ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent, asymptotically Poisson stable) with respect to the variable $t \in \mathbb{R}$ uniformly with respect to x on compact subsets from W, if for every compact $Q \subseteq W$ the function f_Q is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent, asymptotically Poisson stable) with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q$ if and only if there exist functions $P, R \in C(\mathbb{R}\times W, E^n)$ such that $f(t,x) = P(t,x)+R(t,x)$ for all $(t,x) \in \mathbb{R} \times W$. In this case the function P is stationary (resp., τ-periodic, almost periodic, recurrent, Poisson stable) with respect to $t \in \mathbb{R}$ uniformly with respect to x on compact subsets from W and $\lim_{t\to+\infty} |\mathbb{R}(t,x)| = 0 \mapsto \lim_{t\to+\infty} |R(t,x)| = 0$ uniformly with respect to x on compact subsets from W.

Definition 3.10. A solution φ of (3.1) one will call compatible in limit, if it is comparable in limit (in the positive direction) with the function $f_Q := f|_{\mathbb{R}\times Q}$, where $Q = \overline{\varphi(\mathbb{R}_+)}$, that is, the motion $\sigma(\cdot, \varphi)$ generated by the function φ in the dynamical system $(C(\mathbb{R}_+, E^n), \mathbb{R}_+, \sigma)$ is comparable in limit with the motion $\sigma(\cdot, f_Q)$ generated by the function f_Q in the dynamical system $(C(\mathbb{R} \times Q, E^n), \mathbb{R}, \sigma)$.

Definition 3.11. A function $\varphi \in C(\mathbb{R}, E^n)$ is called bounded on $S \subseteq \mathbb{R}$, if the set $\varphi(S) \subset E^n$ is bounded.

Theorem 3.2.1. *Let φ be a bounded on $\mathbb{R}_+$ compatible in limit solution of* (3.1) *and $Q := \overline{\varphi(\mathbb{R}_+)}$. Then:*

(1) *if the right-hand side f of* (3.1) *is asymptotically recurrent with respect to the variable $t \in \mathbb{R}$ uniformly with respect to $x \in Q$, then the solution φ is asymptotically recurrent;*
(2) *if the right-hand side f is asymptotically almost periodic with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q$, then the solution φ is asymptotically almost periodic;*
(3) *if f is asymptotically τ-periodic with respect to to $t \in \mathbb{R}$ uniformly with respect to $x \in Q$, then the solution φ is asymptotically τ-periodic;*
(4) *if f is asymptotically stationary with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q$, then the solution φ is asymptotically stationary.*

Proof. The validity of the formulated statement it follows from the corresponding definitions, [92, Lemma 3.1.1] and Theorem 2.2.2 applied to the nonautonomous dynamical system from Example 3.4. □

Along with (3.1) let us consider the family of "ω-limit" equations

$$\frac{dv}{dt} = g(t, v), \qquad (g \in \omega_f), \tag{3.6}$$

where $f \in C(\mathbb{R} \times W, E^n)$ and ω_f is a ω-limit set of the function f in the dynamical system $(C(\mathbb{R} \times W, E^n), \mathbb{R}, \sigma)$.

Theorem 3.2.2. *Let φ be a bounded on $\mathbb{R}_+$ solution of* (3.1) *and $f_Q := f|_{\mathbb{R}\times Q}$ is st. L^+, where $Q := \overline{\varphi(\mathbb{R}_+)}$. If every equation of family* (3.6) *admits at most one solution from ω_φ, then φ is compatible in limit.*

Proof. Since f_Q is st. L^+, then according to [92, Lemma 3.1.6] the solution φ is st. L^+. Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ be a nonautonomous dynamical system constructed in Example 3.4. Consider an operator equation

$$h(\psi, f_Q) = f_Q. \tag{3.7}$$

Along with (3.7) consider the family of equations

$$h(\psi, g_Q) = g_Q, \qquad (g_Q \in \omega_{f_Q}). \tag{3.8}$$

From the said above it follows that the point $(\varphi, f_Q) \in X$ is st. L_+. According to the conditions of Theorem 3.2.2, every equation of family (3.8) has at least one solution from $\omega_{(\varphi, f_Q)}$. From Theorem 2.3.1 it follows that $\mathfrak{L}^{+\infty}_{f_Q} \subseteq \mathfrak{L}^{+\infty}_{(\varphi, f_Q)}$. To finish the proof of Theorem 3.2.2 it remains to note that $\mathfrak{L}^{+\infty}_{(\varphi, f_Q)} = \mathfrak{L}^{+\infty}_{\varphi} \cap \mathfrak{L}^{+\infty}_{f_Q}$ and, consequently, $\mathfrak{L}^{+\infty}_{f_Q} \subseteq \mathfrak{L}^{+\infty}_{\varphi}$. The theorem is proved. □

Remark 3.12. According to [92, Lemma 3.4.2], the function f_Q is st. L^+ (resp., L) if and only if the following conditions are fulfilled:

(1) f_Q is bounded on $\mathbb{R}_+ \times Q$ (resp., $\mathbb{R} \times Q$), that is, there exists $M > 0$ such that $|f(t, x)| \leq M$ for all $(t, x) \in \mathbb{R}_+ \times Q$ (resp., $(t, x) \in \mathbb{R} \times Q$);
(2) the function f_Q is uniformly continuous on $\mathbb{R}_+ \times Q$ (resp., $\mathbb{R} \times Q$).

Note that every theorem on the compatibility in limit of a solution of (3.1) together with Theorem 3.2.1 gives various tests of existence of asymptotically stationary (resp., asymptotically periodic, asymptotically almost periodic, asymptotically recurrent) solutions of (3.1).

For example from Theorem 3.2.2 follow the next statements.

Corollary 3.13. *Let f be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) with respect to the variable $t \in \mathbb{R}$ uniformly with respect to $x \in Q := \overline{\varphi(\mathbb{R}_+)}$ and φ be a bounded on $\mathbb{R}_+$ solution of* (3.1)*. If every equation of family* (3.6) *admits at most one solution from ω_φ, then φ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).*

Corollary 3.14. *Let φ be a bounded on $\mathbb{R}_+$ solution of* (3.1) *and $f \in C(\mathbb{R}\times W, \mathbb{R})$ ($W \subseteq \mathbb{R}$) be asymptotically recurrent with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q = \overline{\varphi(\mathbb{R}_+)}$. If some function g_Q from ω_{f_Q} ($f_Q := f|_{\mathbb{R}\times Q}$) is strictly monotone with respect to x uniformly with respect to time t, then φ is compatible in limit in the positive direction.*

Proof. From the asymptotical recurrence of f and strict monotonicity of $g_Q \in \omega_{f_Q}$ it follows that every function from ω_{f_Q} possesses the property of strict monotonicity with respect to $x \in Q$ uniformly with respect to $t \in \mathbb{R}$. According to [119], every equation of family (3.8) admits at most one solution from ω_φ and by Theorem 3.2.2 φ is compatible in limit. □

Corollary 3.15. *Let φ be a bounded on $\mathbb{R}_+$ solution of* (3.1) *and $f \in C(\mathbb{R} \times W, \mathbb{R})$ be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q$. If some function $g_Q \in \omega_{f_Q}$ is strictly monotone with respect to $x \in Q$ uniformly with respect to $t \in \mathbb{R}$, then φ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).*

Note that if f is asymptotically almost periodic, then Corollary 3.14 reinforces the result of the work [36].

3.3. Linear Differential Equations

In this section we study linear differential equations satisfying the condition of Favard. We establish the relation between the condition of Favard and regularity and weak regularity. We study also the linear differential equations with asymptotically almost periodic coefficients. Denote by $[E^n]$ the Banach space of all linear bounded operators A acting on E^n with operator norm and by $C_b(I, E^n)$ the Banach space of all continuous and bounded functions $f : I \mapsto E^n$ with sup-norm, where $I \subseteq \mathbb{R}$ is an interval from $\mathbb{R}$ (i.e., $I = [a,b], [a,b), (a,b]$ or (a,b) and $a, b \in \mathbb{R}^* := \mathbb{R} \bigcup \{-\infty, +\infty\}$.

Let **E** and **F** be a pair of subspaces from $C_b(I; E^n)$.

Definition 3.16. An equation

$$\frac{dx}{dt} = A(t)x, \tag{3.9}$$

or differential operator

$$(L_A x)(t) = \frac{dx(t)}{dt} - A(t)x(t), \tag{3.10}$$

where $A \in C(\mathbb{R}; [E^n])$, is called (see, e.g., [107]) $(\mathbf{E}, \mathbf{F})$-admissible (resp., regular), if for every $f \in \mathbf{F}$ the equation

$$L_A x = f \tag{3.11}$$

has at least one (resp., exactly one) solution $\varphi \in \mathbf{E}$.

Definition 3.17. If L_A is $(C_b(\mathbb{R}; E^n), C_b(\mathbb{R}; E^n))$ regular (resp., admissible), then one simply will say that L_A is regular (resp., admissible or weakly regular).

Definition 3.18. Recall that a linear bounded operator $P : E^n \to E^n$ is called a projection, if $P^2 = P$, where $P^2 := P \circ P$.

Definition 3.19. Let $U(t, A)$ be the operator of Cauchy (a solution operator) of linear (3.9). Following [120] one will say that (3.9) has an exponential dichotomy (is hyperbolic) on $I \subseteq \mathbb{R}$, if there exists a projection $P(A) \in [E^n]$ satisfying the following conditions:

(1) $P(A)U(t, A) = U(t, A)P(A)$ for all $t \in I$;
(2) there exist constants $\nu > 0$ and $N > 0$ such that

$$||U_P(t, \tau; A)|| \le N e^{-\nu(t-\tau)} \quad (\forall t \ge \tau;\ t, \tau \in I), \tag{3.12}$$

$$||U_Q(t, \tau; A)|| \le N e^{\nu(t-\tau)} \quad (\forall t \le \tau:\ t, \tau \in I), \tag{3.13}$$

where $U_P(t, \tau; A) := U(t, A)P(A)U^{-1}(\tau, A)$, $U_Q(t, \tau; A) := U(t, A)Q(A)U^{-1}(\tau, A)$ and $Q(A) := E - P(A)$ (E is the identity operator in $[E^n]$).

Theorem 3.3.1 (see [120]). *Let $A \in C_b(\mathbb{R}_+, [E^n])$. Equation* (3.9) *satisfies the condition of exponential dichotomy on $\mathbb{R}_+$ if and only if for every function $f \in C_b(\mathbb{R}_+, E^n)$* (3.10) *admits at least one solution $\varphi \in C_b(\mathbb{R}_+, E^n)$.*

Theorem 3.3.2 (see [107, 120]). *Let $A \in C_b(\mathbb{R}, [E^n])$. Equation* (3.9) *satisfies the condition of exponential dichotomy on $\mathbb{R}$ if and only if for every function $f \in C_b(\mathbb{R}, E^n)$* (3.10) *admits a unique solution $\varphi \in C_b(\mathbb{R}, E^n)$.*

Theorem 3.3.3 (see [120]). *Let $A \in [E^n]$. Equation* (3.10) *admits a unique solution $\varphi \in C_b(\mathbb{R}, E^n)$ for every function $f \in C_b(\mathbb{R}, E^n)$ if and only if the spectrum $\sigma(A)$ of the operator A does not intersect the imaginary axis, that is, $\sigma(A) \cap i\mathbb{R} = \emptyset$, where $i^2 = 1$, $i\mathbb{R} := \{ix \mid x \in \mathbb{R}\}$ and $\sigma(A)$ is the spectrum of operator A.*

3.3.1. Equations with Periodic Operator-Function

Lemma 3.20. *Let* $A \in [E^n]$ *and*

$$L_A x = \frac{dx}{dt} - Ax. \tag{3.14}$$

For the differential operator (3.14) *to be regular it is necessary and sufficient that the equation*

$$L_A x = 0 \tag{3.15}$$

would have no nonzero, bounded on $\mathbb{R}$ *solutions.*

Proof. If operator (3.14) is regular, then, obviously, (3.15) has no nonzero, bounded on $\mathbb{R}$ solutions.

Inversely. Let (3.15) have no nonzero solutions from $C_b(\mathbb{R}, E^n)$. Then the spectrum of the operator A does not intersect with the imaginary axis. In fact, if $i\beta \in \sigma(A)$ $(\beta \in \mathbb{R})$, then there exists $x_0 \in E^n$ $(x_0 \neq 0)$ such that

$$x(t) = e^{i\beta t} x_0 \quad (t \in \mathbb{R}) \tag{3.16}$$

is a nonzero, bounded on $\mathbb{R}$ solution of (3.15) and this contradicts to the condition. So, the spectrum of the operator A does not intersect the imaginary axis. According to Theorem 3.3.3 the operator L_A is regular.

Let us consider an equation of the type

$$\frac{dx}{dt} = A(t)x, \tag{3.17}$$

in which the operator $A(t)$ is a τ-periodic operator-function, that is, for some $\tau > 0$

$$A(t+\tau) = A(t) \quad (t \in \mathbb{R}). \tag{3.18}$$

The Cauchy operator $U(t)$ of (3.17) satisfies the system

$$\begin{gathered} U'(t) = A(t)U(t), \\ U(0) = E. \end{gathered} \tag{3.19}$$

It is easy to see that the operator

$$U_1(t) = U(t+\tau)U^{-1}(\tau) \tag{3.20}$$

satisfies the same system. In virtue of the uniqueness of the solution of system (3.19) we have $U_1(t) = U(t)$, which implies $U(t+\tau) = U(t)U(\tau)$.

Definition 3.21. The operator $U(\tau)$ is called monodromy operator of (3.17).

Since the operator $U(\tau)$ is reversible, then there exists the operator $\mathcal{S} := \ln U(\tau)$ for which $U(\tau) = e^{\mathcal{S}}$.

Let us introduce and consider the operator

$$\Gamma := \frac{1}{\tau} \ln U(\tau). \tag{3.21}$$

Then $U(\tau) = e^{\tau\Gamma}$.

Assume now

$$Q(t) := (t)e^{-t\Gamma}. \tag{3.22}$$

The operator-function Q(t) is τ-periodic:

$$Q(t+\tau) = U(t+\tau)e^{-(t+\tau)\Gamma} = U(t)U(\tau)e^{-t\Gamma} = U(t)e^{-\Gamma t} = Q(t). \tag{3.23}$$

On the segment $[0, \tau]$ this operator-function is continuous, differentiable, and has continuous inverse operator $Q^{-1}(t)$.

From equality (3.22) it follows the Floquet presentation of the Cauchy operator

$$U(t) = Q(t)e^{t\Gamma} \tag{3.24}$$

in the form of the product of periodic differentiable operator-function $Q(t)$ having bounded inverse operator $Q^{-1}(t)$ by the operator exponent $e^{t\Gamma}$ with the constant operator Γ. □

From the said above the theorem on the presentation of Floquet follows.

Theorem 3.3.4 (see [120]). *The operator of Cauchy $U(t)$ of* (3.17) *admits presentation* (3.24), *where $Q(t)$ is a periodic differentiable operator-function having bounded inverse operator $Q^{-1}(t)$, and Γ is a constant operator.*

Definition 3.22. The operator $L \in C(\mathbb{R}; [E^n])$ is called an operator-function of Lyapunov, if there are fulfilled the next conditions:

(1) $L(t)$ and $\dot{L}(t)$ are bounded on $\mathbb{R}$;
(2) $L(t)$ has a bounded inverse operator $L^{-1}(t)$ and the operator-function $L^{-1}(t)$ is bounded on $\mathbb{R}$.

Definition 3.23. The linear transformation

$$x(t) = L(t)y(t), \tag{3.25}$$

with the operator-function of Lyapunov $L(t)$ is called a transformation of Lyapunov.

Definition 3.24. Equation (3.17) is called reducible, if with the help of some transformation of Lyapunov it can be transformed into linear equation

$$\frac{dy}{dt} = By, \tag{3.26}$$

with the constant operator B.

Theorem 3.3.5 (see [120]). *Let the operator-function $A(t)$ be τ-periodic. Then* (3.17) *is reducible.*

Proof. Let us show that $Q(t)$ defined by equality (3.22) is an operator-function of Lyapunov. In fact, since $Q(t)$ and $Q^{-1}(t)$ are continuous with respect to $t \in \mathbb{R}$ and τ-periodic, they are bounded on $\mathbb{R}$. Now we will show that dQ/dt is also bounded on $\mathbb{R}$. For this aim we note that

$$\begin{aligned} \frac{dQ}{dt} &= \frac{d}{dt}\left(U(t)e^{-\Gamma t}\right) = \frac{dU}{dt}e^{-\Gamma t} + U(t)\frac{d}{dt}e^{-\Gamma t} \\ &= A(t)U(t)e^{-\Gamma t} + U(t)\left(-\Gamma e^{-\Gamma t}\right) = A(t)Q(t) - Q(t)\Gamma. \end{aligned} \tag{3.27}$$

From equality (3.27) it follows that dQ/dt is bounded with respect to $t \in \mathbb{R}$.

In (3.17) we make a transformation

$$x(t) = Q(t)y(t). \tag{3.28}$$

Then

$$\frac{dx}{dt} = \frac{dQ}{dt}y(t) + Q(t)\frac{dy}{dt} = \left[A(t)Q(t) - Q(t)\Gamma\right]y + Q(t)\frac{dy}{dt}, \tag{3.29}$$

consequently,

$$A(t)Q(t)y = A(t)Q(t) - Q(t)\Gamma y + Q(t)\frac{dy}{dt}. \tag{3.30}$$

Therefore,

$$\frac{dy}{dt} = \Gamma y. \tag{3.31}$$

The theorem is proved. □

Theorem 3.3.6. *Let $A(t+\tau) = A(t)$ $(t \in \mathbb{R})$. For the equation*

$$\frac{dx}{dt} = A(t)x + f(t) \tag{3.32}$$

to have a unique solution bounded on $\mathbb{R}$ for any bounded on $\mathbb{R}$ function f, it is necessary and sufficient that the spectrum of the operator $\Gamma := (1/\tau)\ln U(t)$ would not intersect the imaginary axis.

Proof. In (3.32) we make the change of the variables $x(t) = Q(t)y(t)$. Then with respect to y we obtain the equation

$$\frac{dy}{dt} = \Gamma y + Q^{-1}(t)f(t), \tag{3.33}$$

from which it follows that (3.32) has a unique bounded on $\mathbb{R}$ solution for any bounded on $\mathbb{R}$ function f exactly when the equation

$$\frac{dy}{dt} = \Gamma y + g(t) \tag{3.34}$$

has single bounded on $\mathbb{R}$ solution for every bounded on $\mathbb{R}$ function $g \in C_b(\mathbb{R}; E^n)$. The last, according to Theorem 3.3.3, takes place if and only if the spectrum of the operator Γ does not cross the imaginary axis. The theorem is proved. □

Corollary 3.25. *Let $A(t)$ be τ-periodic. The operator*

$$L_A x = \frac{dx}{dt} - A(t)x \tag{3.35}$$

is regular if and only if the spectrum of the operator Γ does not intersect the imaginary axis.

Remark 3.26. Note that the spectrum of the operator Γ does not intersect the imaginary axis if and only if the spectrum of the operator of monodromy $U(\tau) = e^{\tau\Gamma}$ does not intersect the unit circle.

Corollary 3.27. *Let $A(t)$ be τ-periodic. Operator* (3.35) *is regular if and only if the equation*

$$L_A x = 0 \tag{3.36}$$

has no nonzero bounded on $\mathbb{R}$ solutions.

Proof. The last statement it follows from Theorem 3.3.5, Lemma 3.20, and the fact that (3.36) and (3.31) at the same time either have or have no bounded on $\mathbb{R}$ solutions. □

Let $A(t+\tau) = A(t)$. Then $H(A) = \{A^{(s)} : s \in [0,\tau)\}$ and, hence, there takes place the next corollary.

Corollary 3.28. *Let $A(t+\tau) = A(t)$. For the operator* (3.35) *to be regular it is necessary and sufficient that every equation*

$$L_B x = 0, \quad (B \in H(A)), \tag{3.37}$$

where $L_B x := (dx/dt) - B(t)x$, would have no nonzero, bounded on $\mathbb{R}$ solutions.

In the next section we generalize the last statement for operators with almost periodic operator-function $A(t)$.

3.3.2. Equations with Almost Periodic Operator-Function

3.3.2.1. Limiting Equations

Let $f \in C(\mathbb{R}; E^n)$.

Definition 3.29. A function $g \in C(\mathbb{R}; E^n)$ is called ω(resp., α)-limit for f, if there exists a sequence $t_n \to +\infty$ (resp., $-\infty$) such that $f^{(t_n)} \to g$ in the topology of the space $C(\mathbb{R}; E^n)$.

By ω_f (resp., α_f) there is denoted the set of all ω (resp., α)-limit functions for f and $\Delta_f := \omega_f \bigcup \alpha_f$. Assume that

$$H^+(f) := \overline{\{f^{(\tau)} : \tau \in \mathbb{R}^+\}}, \qquad H^-(f) := \overline{\{f^{(\tau)} : \tau \in \mathbb{R}^-\}},$$
$$H(f) := H^+(f) \cup H^-(f), \tag{3.38}$$

where by bar there is denoted the closure in the topology of the space $C(\mathbb{R}; E^n)$.

Let$A \in C(\mathbb{R}; [E^n])$. Consider a differential equation

$$\frac{dx}{dt} = A(t)x. \tag{3.39}$$

Denote by $U(t, A)$ the operator of Cauchy of (3.39) and

$$\varphi(t, A, x) = U(t, A)x. \tag{3.40}$$

Lemma 3.30. *The function $U(t, A)$ is continuous with respect to $A \in C(\mathbb{R}; [E^n])$ uniformly with respect to t on compact subsets from $\mathbb{R}$.*

Proof. Let $\{A_n\} \subseteq \mathbb{C}(\mathbb{R}; [E^n])$, $A_n \to A$ uniformly on compact subsets from $\mathbb{R}$ and $l > 0$. Then there exists a number $M(l) > 0$ such that

$$\max_{|t| \le l} ||A_n(t)|| \le M(l) \quad (n \in N). \tag{3.41}$$

Since $U(t, A_n)$ is the solution of the system

$$\dot{U}(t, A_n) = A_n(t)U(t, A_n),$$
$$U(0, A_n) = E, \tag{3.42}$$

then from $\|U(t, A)\| \le \exp\{\int_{t_0}^{t} \|A(t)\| ds\}$ it follows that

$$\max_{|t| \le l} ||U(t, A_n)|| \le e^{2lM(l)} \quad (n \in \mathbb{N}). \tag{3.43}$$

Assume now that $V_n(t) := U(t, A) - U(t, A_n)$ and note that $V_n(t)$ satisfies to the system

$$V_n'(t) = A(t)V_n(t) + [A(t) - A_n(t)]U(t, A_n),$$
$$V_n(0) = 0, \tag{3.44}$$

therefore,

$$V_n(t) = U(t, A) \int_0^t U^{-1}(\tau, A)[A(\tau) - A_n(\tau)]U(\tau, A_n) d\tau. \tag{3.45}$$

Let $K(l) := \max_{|t| \le l}\{\|U(t, A)\|, \|U^{-1}(t, A)\|\}$. From (3.43) and (3.45) it follows the inequality

$$\max_{|t| \le l} ||V_n(t)|| \le K^2(l) 2\, l e^{2lM(l)} \max_{|t| \le l} ||A(t) - A_n(t)||. \tag{3.46}$$

Passing to the limit in (3.46) as $n \to +\infty$ we obtain

$$\lim_{n\to+\infty} \max_{|t|\le l} ||U(t,A) - U(t,A_n)|| = 0. \tag{3.47}$$

The lemma is proved. □

Corollary 3.31. *For every fixed $t \in \mathbb{R}$ the mapping $U_t : C(\mathbb{R}; [E^n]) \to [E^n]$ defined by the equality $U_t(A) := U(t,A)$ is continuous.*

Lemma 3.32. *If* (3.39) *is hyperbolic on $\mathbb{R}_+$, then every equation*

$$\dot{y} = B(t)y, \tag{3.48}$$

where $B \in \omega_A$ is hyperbolic on $\mathbb{R}$.

Proof. Let $B \in \omega_A$. Then there exists $t_n \to +\infty$ such that $B = \lim_{n\to+\infty} A^{(t_n)}$. Assume

$$P(A^{(t_n)}) = U(t_n, A)P(A)U^{-1}(t_n, A), \tag{3.49}$$

$$Q(A^{(t_n)}) = U(t_n, A)Q(A)U^{-1}(t_n, A), \tag{3.50}$$

where $P(A)$ and $Q(A)$ is a pair of mutually complimentary projectors from the definition of exponential dichotomy. Projectors $\{P(A^{(t_n)})\}$ and $\{Q(A^{(t_n)})\}$ are uniformly bounded and, hence, they can be considered convergent. Put $P(B) := \lim_{n\to+\infty} P(A^{(t_n)})$ and $Q(B) := \lim_{n\to+\infty} Q(A^{(t_n)})$. Note that

$$P^2(A^{(t_n)}) = P(A^{(t_n)}), \tag{3.51}$$

$$P(A^{(t_n)}) + Q(A^{(t_n)}) = E. \tag{3.52}$$

Passing to the limit in (3.51) as $t \to +\infty$ we get $P^2(B) = P(B)$. In the same way we show that $Q^2(B) = Q(B)$. At last, from (3.52) it follows that $P(B) + Q(B) = E$. So, $P(B)$ and $Q(B)$ is a pair of mutually complimentary projectors. Let us show that they can be taken as projectors in the definition of the exponential dichotomy on $\mathbb{R}$ of (3.48). In fact, let $t \ge \tau$ and $t, \tau \in \mathbb{R}$. Then for t_n large enough the numbers t and τ belong to the interval $(-t_n, +\infty)$. From the equality

$$U(t, A^{(t_n)})P(A^{(t_n)})U^{-1}(\tau, A^{(t_n)}) = U(t + t_n, A)P(A)U^{-1}(\tau + t_n, A) \tag{3.53}$$

and inequality (3.12), taking into consideration the above said and Lemma 3.30, we obtain the inequality

$$||U(t,B)P(B)U^{-1}(\tau,B)|| \le Ne^{-\nu(t-\tau)}. \tag{3.54}$$

Similarly we prove that

$$||U(t,B)Q(B)U^{-1}(\tau,B)|| \le Ne^{\nu(t-\tau)} \tag{3.55}$$

when $t \le \tau$ and $t, \tau \in \mathbb{R}$. The lemma is proved. □

Corollary 3.33. *Let* (3.39) *be hyperbolic on* $\mathbb{R}_+$ *and* $B \in \omega_A$. *The* (3.48) *has no nonzero bounded on* $\mathbb{R}$ *solutions.*

Corollary 3.34. *Let* (3.39) *be hyperbolic on* $\mathbb{R}_+$ *and* A *be Poisson stable in the positive direction [92], that is,* $A \in \omega_A$. *Then* (3.39) *is hyperbolic on* $\mathbb{R}$.

Lemma 3.35. *Let* (3.39) *be hyperbolic on* $\mathbb{R}$ *and* $B \in H(A)$. *Then* (3.48) *is also exponential dichotomic on* $\mathbb{R}$.

Corollary 3.36. *Let* (3.39) *be hyperbolic on* $\mathbb{R}$ *and* $B \in H(A)$. *Then* (3.48) *has no nonzero bounded on* $\mathbb{R}$ *solutions.*

Naturally the question whether the statement inverse to Corollary 3.36 is true or not arises.

As the example below shows, in general case it is not. In fact, the scalar equation

$$\frac{dx}{dt} = (\arctan t)x \tag{3.56}$$

does not nonzero solutions from $C_b(\mathbb{R}, \mathbb{R})$ and for every $b \in H(a) = \{\arctan(t + \tau) : \tau \in \mathbb{R}\} \bigcup \{\pi/2, -\pi/2\}$ the equation

$$\frac{dx}{dt} = b(t)x \tag{3.57}$$

also does not have nonzero solutions from $C_b(\mathbb{R}, \mathbb{R})$ and however (3.56) be not hyperbolic on $\mathbb{R}$. In fact, if it was not so, then

$$E^s + E^u = E^n, \tag{3.58}$$

where $E^s := \{x \in E^n \mid \lim_{t\to+\infty} |\varphi(t, x, a)| = 0\}$, $E^u := \{x \in E^n \mid \lim_{t\to-\infty} |\varphi(t, x, a)| = 0\}$ and $\varphi(t, x, a) := x \exp(\int_0^t a(s)ds)$ (in our example $a(t) := \arctan t$).

But in our example $E^n = \mathbb{R}$ ($n = 1$), $E^s = \{0\}$ and $E^u = \{0\}$. Hence, (3.58) takes no place.

Note that (3.56) is hyperbolic on $\mathbb{R}_+$ and on $\mathbb{R}_-$. We can easily check it using Theorem 3.3.1.

3.3.2.2. Criterion of Regularity of Almost Periodic Differential Operators

The following lemma takes place.

Lemma 3.37 (see [92]). *Let* $\{\varphi_n\}$, $\{f_n\} \subseteq C(\mathbb{R}; E^n)$, *and* $\{A_n\} \subseteq C(\mathbb{R}; [E^n])$. *If for each* $n \in \mathbb{N}$ *the function* φ_n *satisfies the differential equation*

$$\frac{dx}{dt} = A_n(t)x + f_n(t), \tag{3.59}$$

$A_n \to A$ *and* $f_n \to f$ *uniformly on compact subsets from* $\mathbb{R}$, *then,*

(1) *if* $\{\varphi_n\}$ *is uniformly bounded on* $\mathbb{R}$, *then it is relatively compact in* $C(\mathbb{R}; E^n)$;
(2) *if* $\varphi_n \to \varphi$ *in the topology* $C(\mathbb{R}; E^n)$, *then* φ *is a solution of the differential equation*

$$\frac{dx}{dt} = A(t)x + f(t); \tag{3.60}$$

(3) *if* $\varphi_n(0) \to x_0$, *then* $\{\varphi_n\}$ *converges in* $C(\mathbb{R}; E^n)$ *and* $\varphi = \lim_{n\to+\infty} \varphi_n$ *is a solution of* (3.60) *with the initial condition* $\varphi(0) = x_0$.

Theorem 3.3.7 (see [121]). *Let the operator function* $A \in \mathbb{C}(\mathbb{R}; [E^n])$ *be almost periodic. The differential operator*

$$L_A x = \frac{dx}{dt} - A(t)x \tag{3.61}$$

is regular, if and only if every equation

$$L_B x = 0 \quad (B \in H(A)) \tag{3.62}$$

does admit nonzero solutions from $C_b(\mathbb{R}; E^n)$.

Proof. The necessity of this statement it follows from Theorem 3.3.2 and Corollary 3.36.

Inversely. Let every (3.62) have no nonzero solutions from $C_b(\mathbb{R}, E^n)$ and $A \in C(\mathbb{R}; [E^n])$ be almost periodic. Then there exists $\omega_i \to +\infty$ $(i \to +\infty)$ such that

$$\lim_{i\to+\infty} \sup_{t\in\mathbb{R}} ||A(t+\omega_i) - A(t)|| = 0. \tag{3.63}$$

Consider differential operators

$$L_{A_i} x = \frac{dx}{dt} - A_i(t)x \tag{3.64}$$

with continuous ω_i-periodic coefficients $A_i(t)$:

$$A_i(t) := \begin{cases} A(t), & \text{if } \dfrac{1}{i} \le t \le \omega_i, \\ it[A(t) - A(\omega_i)] + A(\omega_i), & \text{if } 0 \le t \le \dfrac{1}{i}. \end{cases} \tag{3.65}$$

Let us show that

$$\lim_{i\to+\infty} \max_{|t|\le\omega_i} ||A_i(t) - A(t)|| = 0. \tag{3.66}$$

In fact,

$$\begin{aligned}
&\max_{|t|\le\omega_i}||A_i(t)-A(t)||\\
&\quad\le \max_{1/i\le t\le\omega_i}||A_i(t)-A(t)||+\max_{-\omega_i+1/i\le t\le 0}||A_i(t)-A(t)||\\
&\qquad+\max_{0\le t+\omega_i\le 1/i}||A_i(t)-A(t)||+\max_{0\le t\le 1/i}||A_i(t)-A(t)||\\
&\quad\le \max_{1/i\le t+\omega_i\le\omega_i}||A(t+\omega_i)-A(t)||+\max_{0\le t+\omega_i\le 1/i}||A_i(t)-A(t)||+\max_{0\le t\le 1/i}||A_i(t)-A(t)||\\
&\quad\le \max_{1/i\le t+\omega_i\le\omega_i}||A(t+\omega_i)-A(t)||+\max_{0\le t+\omega_i\le 1/i}||A_i(t)-A(t)||+\max_{0\le t\le 1/i}||A_i(t)-A(t)||.
\end{aligned} \tag{3.67}$$

Besides,

$$\begin{aligned}
&\max_{0\le t\le 1/i}||A_i(t)-A(t)||\\
&\quad=\max_{0\le t\le 1/i}||it[A(t)-A(\omega_i)]+A(\omega_i)-A(t)||\\
&\quad=\max_{0\le t\le 1/i}||(it-1)A(t)+(-it+1)A(\omega_i)||=\max_{0\le t\le 1/i}||(-it+1)[A(t)-A(\omega_i)]||\\
&\quad\le 2\max_{0\le t\le 1/i}||A(t)-A(0)||+2||A(0)-A(\omega_i)||,
\end{aligned} \tag{3.68}$$

$$\begin{aligned}
&\max_{0\le t+\omega_i\le 1/i}||A_i(t)-A(t)||\\
&\quad=\max_{0\le t+\omega_i\le 1/i}||A_i(t+\omega_i)-A(t+\omega_i)+A(t+\omega_i)-A(t)||\\
&\quad\le\max_{0\le t+\omega_i\le 1/i}||A_i(t+\omega_i)-A(t+\omega_i)||+\max_{0\le t+\omega_i\le 1/i}||A(t+\omega_i)-A(t)||\\
&\quad\le\max_{0\le t\le 1/i}||A_i(t)-A(t)||+\sup_{t\in\mathbb{R}}||A(t+\omega_i)-A(t)||.
\end{aligned} \tag{3.69}$$

From (3.63) and (3.67)–(3.69) it follows equality (3.66).

Let us show that each of operators L_{A_i} $(i\in\mathbb{N})$ is reversible and their norms $\|L_{A_i}^{-1}\|$ are uniformly bounded. For this aim we note that there exists $\delta>0$ such that

$$||L_{A_i}\varphi||\ge\delta\|\varphi\|\quad(i\in\mathbb{N}). \tag{3.70}$$

Let us suppose the contrary. Then there exists $\{\varphi_k\}\subseteq C_b(\mathbb{R};E^n)$ ($\|\varphi_k\|=1$ for every $k\in\mathbb{N}$) and $\delta_k\to 0$ ($\delta_k>0$) such that

$$||L_{A_{i_k}}\varphi_k||\le\delta_k\quad(k\in\mathbb{N}). \tag{3.71}$$

Since $\|\varphi_k\|=\sup\{|\varphi_k(t)|:t\in\mathbb{R}\}=1$, then there exists $t_k\in\mathbb{R}$ such that

$$|\varphi_k(t_k)|\ge\frac{1}{2}\quad(k\in\mathbb{N}). \tag{3.72}$$

Let us present t_k in the form $t_k=n_k\omega_{i_k}+\tau_k$, where $|\tau_k|\le(1/2)\omega_{i_k}$.

Denote by ψ_k the functions defined by the equality

$$\psi_k(t) = \varphi_k\,(t + n_k\omega_k) \quad (t \in \mathbb{R}). \tag{3.73}$$

Note that

$$||\psi_k|| = ||\varphi_k|| = 1, \tag{3.74}$$

$$\begin{aligned}(L_{A_k}\psi_k)(t) &= \psi_k{}'(t) - A_k(t)\psi_k(t)\\ &= \varphi_k{}'(t + n_k\omega_k) - A_k(t + n_k\omega_k)\varphi(t + n_k\omega_k) = (L_{A_k}\varphi_k)(t + n_k\omega_k).\end{aligned} \tag{3.75}$$

From (3.71) and (3.75) it follows that

$$||L_{A_k}\psi_k|| \le \delta_k \quad (k \in \mathbb{N}). \tag{3.76}$$

So, we found a sequence $\{\tau_k\} \subseteq [-\omega_{i_k}/2,\ \omega_{i_k}/2]$ such that

$$|\,\psi_k(\tau_k)\,| \ge \frac{1}{2}, \qquad ||\psi_k|| = 1, \qquad ||L_{A_k}\psi_k|| \le \delta_k. \tag{3.77}$$

Consider a sequence $B_k(t) := A(t+\tau_k)$. In virtue of the almost periodicity of the operator-function $A(t)$ the sequence $\{B_k\}$ can be considered convergent in $C_b(\mathbb{R}; [E^n])$. Assume $B := \lim_{k\to+\infty} B_k$. Obviously $B \in H(A)$. Let us construct the sequence $\widetilde{A}_k(t) := A_{i_k}(t+\tau_k)$. We will show that there takes place the equality

$$\lim_{k\to+\infty}\ \max_{|t|\le\omega_{i_k}/2} ||A_{i_k}(t + \tau_k) - A(t + \tau_k)|| = 0. \tag{3.78}$$

In fact, if $|t| \le \omega_{i_k}/2$, then $|t + \tau_k| \le \omega_{i_k}$ and, consequently,

$$\max_{|t|\le\omega_{i_k}/2} ||A_{i_k}(t + \tau_k) - A(t + \tau_k)|| \le \max_{|s|\le\omega_k} ||A_{i_k}(s) - A(s)||. \tag{3.79}$$

Passing to the limit as $k \to +\infty$ in (3.79) and taking into consideration (3.66), we obtain (3.78). Then

$$\begin{aligned}&\max_{|t|\le\omega_{i_k}/2} ||A_{i_k}(t + \tau_k) - B(t)||\\ &\quad\le \max_{|t|\le\omega_{i_k}/2} ||A_{i_k}(t + \tau_k) - A(t + \tau_k)|| + \max_{|t|\le\omega_{i_k}/2} ||A(t + \tau_k) - B(t)||,\end{aligned} \tag{3.80}$$

consequently, $A_{i_k}(t + \tau_k) \to B(t)$ in the topology of the space $C(\mathbb{R}; [E^n])$. Put

$$f_k(t) := \psi_k{}'(t + \tau_k) - \widetilde{A}_k(t)\psi_k(t + \tau_k) \tag{3.81}$$

for all $t \in \mathbb{R}$. From (3.77) it follows that $\|f_k\| \to 0$.

Let us consider the differential equation

$$\frac{dx}{dt} = \widetilde{A}_k(t)x + f_k(t), \tag{3.82}$$

It is obvious that $\psi_k(t+\tau_k)$ is a solution of (3.82) and $|\psi_k(t+\tau_k)| \leq 1$ for all $t \in \mathbb{R}$. According to Lemma 3.37, the sequence $\{\psi_k(t+\tau_k)\}$ can be considered convergent. Put $\psi(t) := \lim_{k\to+\infty} \psi_k(t+\tau_k)$. By the same lemma

$$\frac{d\psi(t)}{dt} = B(t)\psi(t) \quad (t \in \mathbb{R}) \tag{3.83}$$

and $\|\psi\| \leq 1$. Besides, from (3.77) it follows that $|\psi(0)| \geq 1/2$ and, consequently, ψ is a nonzero solution from $C_b(\mathbb{R}; E^n)$ of (3.83), and $B \in H(A)$. The last contradicts to the condition of the theorem. The obtained contradiction shows the required statement.

So, there exists $\delta > 0$ such that (3.70) holds. Obviously, every equation

$$L_{A_i} x = 0 \tag{3.84}$$

has no nonzero solutions from $C_b(\mathbb{R}; E^n)$. According to Corollary 3.27 there exists $L_{A_i}^{-1}$. From inequality (3.70) it follows that

$$||L_{A_i}^{-1}|| \leq \delta^{-1} \quad (i \in \mathbb{N}). \tag{3.85}$$

Let now $f \in C_b(\mathbb{R}; E^n)$. Consider nonhomogeneous equations

$$L_{A_i} x = f \quad (i \in \mathbb{N}). \tag{3.86}$$

Every of these equations has a unique solution φ_i from $C_b(\mathbb{R}; E^n)$, and

$$||\varphi_i|| \leq \delta^{-1} \|f\|. \tag{3.87}$$

According to Lemma 3.37, the sequence $\{\varphi_i\}$ is relatively compact in $C(\mathbb{R}; E^n)$. To be simple, we consider it convergent and put $\varphi := \lim_{i\to+\infty} \varphi_i$. By Lemma 3.37

$$\frac{d\varphi(t)}{dt} = A(t)\varphi(t) + f(t), \tag{3.88}$$

and $|\varphi(t)| \leq \delta^{-1}\|f\|$ for all $t \in \mathbb{R}$. So, for every $f \in C_b(\mathbb{R}; E^n)$ (3.88) has a unique solution in $C_b(\mathbb{R}; E^n)$. The theorem is proved. □

3.3.3. Equations Satisfying the Condition of Favard in the Positive Direction

3.3.3.1. General Definitions

Let us consider equations

$$\frac{dx}{dt} = A(t)x, \tag{3.89}$$

$$\frac{dy}{dt} = B(t)y, \tag{3.90}$$

where A and $B \in C(\mathbb{R}; [E^n])$.

Definition 3.38. One says that (3.89) satisfies the condition of Favard in the positive (resp., negative) direction, denoting it by Φ^+(resp., Φ^-), if for every $B \in \omega_A$(resp., α_A) (3.90) has no nonzero bounded on $\mathbb{R}$ solutions.

Definition 3.39. If (3.89) satisfies the condition Φ^+ and Φ^-, then one say that (3.89) satisfies the two-sided (or weak condition of Favard) and denote it Φ.

Definition 3.40. If for each $B \in H(A)$ (3.90) has no nonzero bounded on $\mathbb{R}$ solutions, then one says that (3.89) satisfies the condition of Favard (or reinforced condition of Favard) and denote it $\mathcal{F}$.

Between the introduced notions there exists a close relation. We study this question below. Here we only note that from the weak condition of Favard, generally speaking, the condition of Favard does not follow (the inverse is obvious). The last is confirmed by the example

$$\frac{dx}{dt} = (\arctan t)x. \tag{3.91}$$

It is easy to check that (3.91) satisfies the condition Φ but does not satisfy the condition $\mathcal{F}$, since (3.91) has nonzero bounded on $\mathbb{R}$ solutions.

Definition 3.41. They say that a function $f \in C(\mathbb{R}; E^n)$ is stable by Lagrange in the positive (resp., negative) direction, denoting it L^+ (resp., L^-), if $H^+(f)$ (resp., $H^-(f)$) is compact in $C(\mathbb{R}; E^n)$.

Definition 3.42. If the function $f \in C(\mathbb{R}; E^n)$ is st. L^+ and st. L^-, then they say that f is stable by Lagrange and denote it st. L.

3.3.3.2. Some Properties of Equations Satisfying the Condition of Favard in the Positive Direction

Lemma 3.43. *Let $A \in C(\mathbb{R}; [E^n])$ be st. L^+ (resp., L^-) and* (3.89) *satisfy the condition Φ^+ (resp., Φ^-). Then, if $\varphi(t, A, x)$ is bounded on $\mathbb{R}_+$ (resp., $\mathbb{R}_-$), then*

$$\lim_{t\to+\infty} |\varphi(t, A, x)| = 0 \quad \Big(\textit{resp.}, \lim_{t\to-\infty} |\varphi(t, A, x)| = 0\Big). \tag{3.92}$$

Proof. Suppose the contrary, that is, there exist $\varepsilon_0 > 0$ and $t_k \to +\infty$ such that $|\varphi(t_k, A, x)| \geq \varepsilon_0$. Without loss of generality we can consider that the sequences $\{\varphi(t_k, A, x)\}$ and $\{A^{(t_k)}\}$ are convergent in E^n and $C(\mathbb{R}; [E^n])$, respectively. Assume $x_0 = \lim_{k\to+\infty} \varphi(t_x, A, x)$ and $B = \lim_{k\to+\infty} A^{(t_k)}$. Then according to Lemma 3.37, $\varphi(t, B, x_0) = \lim_{k\to+\infty}(t + t_k, A^{(t_k)}, x)$ is a nontrivial bounded on $\mathbb{R}$ solution of (3.90) and $B \in \omega_A$. The last contradicts to the condition of Lemma 3.43. The second case is considered in the similar way. □

Lemma 3.44. *Let $A \in C(\mathbb{R}; [E^n])$ be st. L^+ and let* (3.89) *satisfy the condition Φ^+. If $\varphi(t, A, x)$ is unbounded on $\mathbb{R}_+$, then* $\lim_{t\to+\infty} |\varphi(t, A, x)| = +\infty$.

Proof. Let us suppose the contrary, that is, for some $L > 0$ there exist sequences $\{s_k\}$, $\{t_k\}$, and $\{l_k\}$ satisfying the next conditions:

(1) $s_k < t_k < l_k < s_{k+1}$;
(2) $\{s_k\} \to +\infty$ as $k \to +\infty$;
(3) $|\varphi(\tau, A, x)| > L$ for all $\tau \in (s_k, l_k)$;
(4) $|\varphi(s_k, A, x)| = |\varphi(l_k, A, x)| = L$;
(5) $|\varphi(t_k, A, x)| = \max\{|\varphi(t, A, x)| : t \in [s_k, l_k]\}$.

Assume

$$x_k := \frac{\varphi(t_k, A, x)}{|\varphi(t_k, A, x)|} = |\varphi(t_k, A, x)|^{-1} \cdot \varphi(t_k, A, x). \tag{3.93}$$

Then there are held the relations:

(a) $|x_k| = 1$;
(b) $|\varphi(t, A^{(t_k)}, x_k)| = |\varphi(t_k, A, x)|^{-1} \cdot |\varphi(t + t_k, A, x)| \leq 1$

for all $t \in [s_k - t_k, l_k - t_k]$. Let us show that $\{s_k - t_k\} \to -\infty$ ($\{l_k - t_k\} \to +\infty$). In fact, under the conditions of Lemma 3.44 the sequences $\{A^{(t_k)}\}$ and $\{x_k\}$ can be considered convergent. Put $x_0 = \lim_{k\to+\infty} x_k$ and $B := \lim_{k\to+\infty} A^{(t_k)}$. If we suppose that $\{s_k - t_k\} \not\to -\infty$, then it can be considered convergent. Assume $\tau_0 := \lim_{k\to+\infty}\{s_k - t_k\}$. Passing to the limit in the equality

$$|\varphi(s_k - t_k, A^{(t_k)}, x_k)| = |\varphi(t_k, A, x)|^{-1} \cdot |\varphi(s_k, A, x)| = L|\varphi(t_k, A, x)|^{-1}, \tag{3.94}$$

we obtain that $\varphi(\tau_0, B, x_0) = 0$. From the last equality it follows that $x_0 = 0$ that contradicts to the condition (a). In the same way it is proved that $\{l_k - t_k\} \to +\infty$. From the said above and the condition (b) it follows that $|\varphi(t, B, x_0)| \leq 1$ holds for all $t \in \mathbb{R}$, and $|x_0| = 1$. So, $\varphi(t, B, x_0)$ is a nontrivial bounded on $\mathbb{R}$ solution of (3.90). The obtained contradiction proves the lemma. □

Lemma 3.45. *Let $A \in C(\mathbb{R}; [E^n])$ be st. L^+ and* (3.89) *satisfy the condition Φ^+. If a solution $\varphi(t, A, x)$ is unbounded on $\mathbb{R}_+$, then there exists $c > 0$ such that*

$$\max_{0\leq t\leq\tau} |\varphi(t, A, x)| \leq c|\varphi(\tau, A, x)| \tag{3.95}$$

for all $\tau \in \mathbb{R}_+$.

Proof. Suppose the contrary. Then for every $k \in \mathbb{N}$ there is $L_k \geq k$ such that

$$\max_{0\leq t\leq L_k} |\varphi(t, A, x)| \geq k|\varphi(L_k, A, x)|. \tag{3.96}$$

Choose $\tau_k \in [0, L_k]$ so that the equality

$$|\varphi(\tau_k, A, x)| = \max_{0\leq t\leq L_k} |\varphi(t, A, x)| \tag{3.97}$$

holds. Then inequality (3.96) takes the form

$$|\varphi(\tau_k, A, x)| \geq k|\varphi(L_k, A, x)|. \tag{3.98}$$

Put

$$x_k := \frac{\varphi(\tau_k, A, x)}{|\varphi(\tau_k, A, x)|} = |\varphi(\tau_k, A, x)|^{-1} \cdot \varphi(\tau_k, A, x). \tag{3.99}$$

Then

$$|x_k| = 1, \qquad |\varphi(t, A^{(\tau_k)}, x_k)| = |\varphi(\tau_k, A, x)|^{-1} \cdot |\varphi(t + \tau_k, A, x)| \tag{3.100}$$

for all $t \in [\tau_k, L_k - \tau_k]$. Reasoning in the same way that in Lemma 3.44 and taking into consideration the relations

(a) $|\varphi(L_k - \tau_k, A^{(\tau_k)}, x_k)| = |\varphi(\tau_k, A, x)|^{-1} \cdot |\varphi(L_k, A, x)| \le k^{-1}$,

(b) $|\varphi(-\tau_k, A^{(\tau_k)}, x)| = |\varphi(\tau_k, A, x)|^{-1} \cdot |x|$,

we obtain that $\{\tau_k\} \to +\infty$ and $\{L_k - \tau_k\} \to +\infty$. Without loss of generality we can consider that the sequences $\{A^{(\tau_k)}\}$ and $\{x_k\}$ are convergent. Let $x_0 := \lim_{k\to+\infty} x_k$ and $B := \lim_{k\to+\infty} A^{(\tau_k)}$. It is clear that $B \in \omega_A$, $|x_0| = 1$ and $|\varphi(t, B, x_0)| \le 1$ for all $t \in \mathbb{R}$. The last contradicts to the condition of the lemma. □

3.3.3.3. Scalar Equations

Below we suppose that the space E^n is one-dimensional ($E^n = \mathbb{R}$), that is, $A = a \in C(\mathbb{R};\mathbb{R})$.

Lemma 3.46. *Let $a \in C(\mathbb{R};\mathbb{R})$ be st. L^+. If* (3.89) *satisfies the condition Φ^+ and the solutions $\varphi(t, a, x)$ of* (3.90) *as $x \ne 0$ are unbounded for $t \in \mathbb{R}^+$. Then solutions of every* (3.90) *with $B = b \in \omega_a$ are bounded on $\mathbb{R}_-$.*

Proof. Let $b \in \omega_a$. Then there exists $t_k \to +\infty$ such that $b = \lim_{k\to+\infty} a^{(t_k)}$. define a sequence

$$x_k := \frac{\varphi(t_k, a, x)}{\alpha_k}, \tag{3.101}$$

where $\alpha_k := \max\{\varphi(t, a, x) : t \in [0, t_k]\}$. For it there are fulfilled the following conditions:

(1) $|x_k| = 1$;

(2) $|\varphi(t, a^{(t_k)}, x_k)| = \alpha_k^{-1} |\varphi(t + t_k, a, x)| \le 1$

for all $t \in [-t_k, 0]$. In virtue of (1) the sequence $\{x_k\}$ can be considered convergent. Assume $x_0 := \lim_{k\to+\infty} x_k$. Passing to the limit in inequality (2) we get $|\varphi(t, b, x_0)| \le 1$ for all $t \in \mathbb{R}_-$. In this case, by Lemma 3.45, $|x_k| \ge c^{-1} > 0$ and, consequently, $|x_0| \ge c^{-1} > 0$. To finish the proof of the lemma it is sufficient to use the fact that (3.89) is a linear homogeneous scalar equation. □

Lemma 3.47. *Let a be st. L^+ and* (3.89) *satisfy the condition Φ^+. If all solutions of* (3.89) *are bounded on $\mathbb{R}_+$, then there exist numbers $N, \nu > 0$ such that*

$$|\varphi(t, b, x)| \le N e^{-\nu t} |x| \tag{3.102}$$

for all $x \in \mathbb{R}$, $b \in H^+(a)$ and $t \in \mathbb{R}_+$.

Proof. The proof of the formulated statement may be done by repeating exactly the reasoning from the work [122]. □

Lemma 3.48. *Let a be st. L^+ and* (3.89) *satisfy the condition* Φ^+. *Then* (3.89) *is hyperbolic on* $\mathbb{R}_+$.

Proof. Logically, two cases are possible:

(a) all solutions of (3.89) are bounded on $\mathbb{R}_+$ and then Lemma 3.48 follows from Lemma 3.47;

(b) all nontrivial solutions of (3.89) are unbounded on $\mathbb{R}_+$. According to Lemma 3.44 when $x \neq 0$ there takes place the equality

$$\lim_{t\to+\infty} |\varphi(t,a,x)| = +\infty, \tag{3.103}$$

hence, in (3.89) we can change of variables: $y = 1/x$. In this case it turns in the next equation

$$\frac{dy}{dt} = -a(t)y. \tag{3.104}$$

Together with (3.104) we consider the family of equations

$$\frac{dx}{dt} = -\widetilde{b}(t)x, \quad (\widetilde{b} \in \omega_{(-a)}). \tag{3.105}$$

Let us show that (3.104) satisfies the conditions of Lemma 3.47. Let $\widetilde{b} \in \omega_{(-a)}$. Then there exists $\widetilde{b} \in \omega_a$ such that $\widetilde{b} = -b$. By Lemma 3.46, every solution of (3.90) is bounded for $t \in \mathbb{R}_-$ and, consequently, according to the condition of the lemma they are unbounded on $t\mathbb{R}_+$. From Lemmas 3.43 and 3.44 it follows that for every $x \neq 0$ and $b \in \omega_a$ there are fulfilled the conditions

$$\lim_{t\to-\infty} |\varphi(t,a,x)| = 0, \tag{3.106}$$

$$\lim_{t\to+\infty} |\varphi(t,a,x)| = +\infty. \tag{3.107}$$

Note that

$$\varphi(t,\widetilde{b},x) = \varphi(t,-b,x) = \frac{1}{\varphi(t,b,1/x)}, \tag{3.108}$$

therefore, from (3.106) and (3.107) it follows that for any $x \neq 0$ and $\widetilde{b} \in \omega_{(-a)}$ there are held the equalities

$$\lim_{t\to+\infty} | \varphi(t,\widetilde{b},x) = 0, \qquad \lim_{t\to-\infty} | \varphi(t,\widetilde{b},x) = +\infty, \tag{3.109}$$

that is, (3.104) satisfies the condition Φ^+. It is not difficult to establish that all solutions of (3.104) are bounded on $\mathbb{R}_+$. Consequently, according to Lemma 3.47 there are numbers

$N > 0$ and $\nu > 0$ such that

$$|\varphi(t, \widetilde{b}, x)| \le N e^{-\nu t}|x| \tag{3.110}$$

for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}$ and $\widetilde{b} \in H^+(-a)$.

Let $x \neq 0$ and $b \in H^+(a)$. Then there exists $\widetilde{b} \in H^+(-a)$ such that $\widetilde{b} = -\widetilde{b}$ and, consequently, there takes place the inequality

$$|\varphi(t, -\widetilde{b}, x)| = |\varphi(t, b, x)|^{-1} \le N e^{-\nu t}|x|^{-1}. \tag{3.111}$$

Last inequality implies that $|\varphi(t, b, x)| \ge N e^{-\nu t}|x|$ for all $x \in \mathbb{R}$, $b \in H^+(a)$ and $t \in \mathbb{R}_+$. And, hence,

$$|\varphi(t, b, x)| = |\varphi(t - \tau, b^{(\tau)}, \varphi(\tau, b, x)| \ge N^{-1} e^{-\nu(t-\tau)} |\varphi(\tau, b, x)| \tag{3.112}$$

for $t \ge \tau$. Therefore,

$$|\varphi(\tau, b, x)| \le N e^{-\nu(t-\tau)} |\varphi(t, b, x)| \tag{3.113}$$

for $\tau \ge t \ge 0$. The lemma is proved. □

3.3.3.4. Equations of General Type

Lemma 3.49. *Let A be st. L^+. If* (3.89) *satisfies the condition of Φ^+ and A is a down-triangular matrix, then* (3.89) *is hyperbolic on $\mathbb{R}_+$.*

Proof. Let us prove it by induction by the dimensionality n of the space E^n. In the case when $n = 1$ the fact that Lemma 3.49 is true follows from Lemma 3.48. Suppose that the lemma is true when $k(k < n)$.

(I) Consider $k + 1$—dimensional system

$$\begin{aligned} x_1' &= a_{11}(t)x_1, \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2, \\ &\dots \\ x_{k+1}' &= a_{k+1,1}(t)x_1 + \cdots a_{k+1,k+1}(t)x_{k+1}. \end{aligned} \tag{3.114}$$

According to Theorem 3.3.1 for (3.89) to be hyperbolic on $\mathbb{R}_+$ it is necessary to show that for every bounded on $\mathbb{R}_+$ function $f \in C_b(\mathbb{R}_+; E^n)$ the equation

$$\frac{dx}{dt} = A(t)x + f(t) \tag{3.115}$$

has at least one solution $\varphi \in C_b(\mathbb{R}_+; E^n)$.

(II) Let us show that the system

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + f_1(t), \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + f_2(t), \\ &\dots \\ x_{k+1}' &= a_{k+1,1}(t)x_1 + \cdots a_{k+1,k+1}(t)x_{k+1} + f_{k+1}(t) \end{aligned} \tag{3.116}$$

has at least one bounded on $\mathbb{R}_+$ solution.

(III) In fact, it is easy to see that the equation

$$x_1' = a_{11}(t)x_1, \tag{3.117}$$

and the system

$$\begin{aligned} x_2' &= a_{22}(t)x_2, \\ x_3' &= a_{32}(t)x_2 + a_{33}(t)x_3, \\ &\dots \\ x_{k+1}' &= a_{k+1,2}(t)x_2 + \cdots a_{k+1,k+1}(t)x_{k+1} \end{aligned} \tag{3.118}$$

satisfy the condition of exponential dichotomy on $\mathbb{R}_+$ by definition. Using this fact it is easy to see that system (3.116) has at least one bounded on $\mathbb{R}_+$ solution. The lemma is proved. □

Lemma 3.50 (see [123]). *Let A be st. L^+. Then*

(1) *there exists a st. L^+ down-triangular matrix P such that* (3.89) *can be reduced to the equation*

$$\frac{dx}{dt} = P(t)x; \tag{3.119}$$

(2) *for any $Q \in \omega_P$ there exists $B \in \omega_A$ such that* (3.90) *can be reduced to the equation*

$$\frac{dx}{dt} = Q(t)x. \tag{3.120}$$

From this lemma directly it follows that.

Corollary 3.51. *For Lyapunov's transformation there are preserved the next properties: exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}$, regularity, the condition Φ^+, Φ^-, and $\mathcal{F}$.*

Theorem 3.3.8. *Let A be st. L^+. For* (3.89) *to be hyperbolic on $\mathbb{R}_+$ it is necessary and sufficient that* (3.89) *would satisfy the condition Φ^+.*

Proof. Let A be st. L^+ and let (3.89) be hyperbolic on $\mathbb{R}_+$. According to Corollary 3.33, (3.89) satisfies the condition Φ^+.

Sufficiency. Let A be st. L^+ and (3.89) satisfy the condition Φ^+. According to Lemma 3.50, (3.89) can be brought to triangular form (3.119) with L^+ stable matrix P. Let us show that (3.119) satisfies the condition of Lemma 3.49. In fact, let $Q \in \omega_P$. By Lemma 3.50 there exists $B \in \omega_A$ such that (3.90) can be reduced to (3.120). Since the relation of reduction is symmetric, then (3.120) has a nontrivial bounded on $\mathbb{R}$ solution if and only if (3.90) has such solutions. So, in our case every (3.120), where $Q \in \omega_P$, has no nontrivial bounded on $\mathbb{R}$ solutions. According to Lemma 3.49, (3.119) is hyperbolic on $\mathbb{R}_+$, and, consequently, (see Corollary 3.51), (3.89) is hyperbolic on $\mathbb{R}_+$. The theorem is proved. □

Remark 3.52. All results of this chapter naturally can be formulated and proved for equations satisfying the condition Φ^+.

3.3.4. Equations Satisfying the Condition of Favard

Let E and F be a pair of subspaces from $C_b(I;E^n)$. Recall that an equation

$$\frac{dx}{dt} = A(t)x, \tag{3.121}$$

or differential operator

$$L_A x = \frac{dx}{dt} - A(t)x, \tag{3.122}$$

where $A \in C(\mathbb{R};[E^n])$, is called (E,F)-admissible (resp., regular), if for every $f \in F$ the equation

$$L_A x = f \tag{3.123}$$

has at least one (resp., exactly one) solution $\varphi \in E$.

If $L_A(C_b(\mathbb{R};E^n),\ C_b(\mathbb{R};E^n))$ is regular (resp., admissible), then we simply will say that L_A is regular (resp., admissible or weakly regular).

From the results of the previous subsections it follows.

Theorem 3.3.9. *Let $A \in C(\mathbb{R};[E^n])$ be st. L.* (3.121) *satisfies the condition Φ, if and only if it is hyperbolic on $\mathbb{R}_+$ and $\mathbb{R}_-$.*

3.3.4.1. Scalar Equations

In this section we suppose that the space E^n is one-dimensional ($E^n = \mathbb{R}$) and $A = a \in C(\mathbb{R};\mathbb{R})$. Let us introduce the following notation:

$$\lambda(t,\tau,a) := \frac{1}{t-\tau}\int_\tau^t a(s)\,ds, \tag{3.124}$$

$$\begin{aligned} \Lambda^+(a) &:= \overline{\lim_{\substack{\tau\to+\infty\\ t-\tau\to+\infty}}}\ \lambda(t,\tau,a), \qquad \lambda^+(a) := \underline{\lim_{\substack{\tau\to+\infty\\ t-\tau\to+\infty}}}\ \lambda(t,\tau,a),\\ \Lambda^-(a) &:= \overline{\lim_{\substack{\tau\to+\infty\\ t-\tau\to+\infty}}}\ \lambda(t,\tau,a), \qquad \lambda^-(a) := \underline{\lim_{\substack{\tau\to+\infty\\ t-\tau\to+\infty}}}\ \lambda(t,\tau,a). \end{aligned} \tag{3.125}$$

Note that $\lambda^+(a) \le \Lambda^+(a)$; $\lambda^-(a) \le \Lambda^-(a)$ and $\Lambda^+(-a) = -\lambda^+(a)$; $\Lambda^-(-a) = -\lambda^-(a)$; $\lambda^+(-a) = -\Lambda^+(a)$; $\lambda^-(-a) = -\Lambda^-(a)$.

Lemma 3.53. *Let a be st. L^+ (resp., L^-). Equation* (3.121) *satisfies the condition Φ^+ (resp., Φ^-), if and only if there holds the inequality $\Lambda^+(a)\lambda^+(a) > 0$ (resp., $\Lambda^-(a)\lambda^-(a) > 0$).*

Proof. Let a be st. L^+ and (3.121) satisfy the condition Φ^+. According to Theorem 3.3.8, (3.121) is hyperbolic on $\mathbb{R}_+$ and, consequently, there exist positive numbers N and ν such that there takes place one of the following inequalities:

$$|\varphi(t,a,x)| \le Ne^{-\nu(t-\tau)}|\varphi(\tau,a,x)| \tag{3.126}$$

for all $t \ge \tau \ge 0$ and $x \in \mathbb{R}$ or

$$|\varphi(t,a,x)| \le Ne^{\nu(t-\tau)}|\varphi(\tau,a,x)| \tag{3.127}$$

for all $t \le \tau \le 0$ and $x \in \mathbb{R}$.

Let inequality (3.126) be fulfilled. Then it is not difficult to see that $\lambda^+(a) \le \Lambda^+(a) \le -\nu$. It implies the inequality $\Lambda^+(a)\lambda^+(a) > 0$. If there takes place inequality (3.127), then $\Lambda^+(a) \ge \lambda^+(a) \ge \nu > 0$ and, consequently $\Lambda^+(a)\lambda^+(a) > 0$.

Inversely. Let a be st. L^+ and $\Lambda^+(a)\lambda^+(a) > 0$. Logically, two cases are possible:

(a) $\Lambda^+(a) \ge \lambda^+(a) > 0$. Assuming

$$\mu(a) := \varliminf_{t\to+\infty} \frac{1}{t}\int_0^t a(s)\,ds, \tag{3.128}$$

let us show that for every function $b \in \omega_a$ the inequality $\mu(b) > 0$ holds. Indeed. Since $\lambda^+(a) > 0$, then for every number $\varepsilon \in (0, \lambda^+(a))$ there exists $T(\varepsilon) > 0$ such that

$$\lambda(t,\tau,a) \ge \lambda^+(a) - \varepsilon \tag{3.129}$$

for all $\tau \ge T(\varepsilon)$ and $t - \tau \ge T(\varepsilon)$. Let $b \in \omega_a$. Then there exists $t_m \to +\infty$ such that $b = \lim_{m\to+\infty} a^{(t_m)}$, and t is an arbitrary fixed number from $[T(\varepsilon), +\infty)$. Hence,

$$\frac{1}{t}\int_0^t a(s+t_m)ds = \frac{1}{t}\int_{t_m}^{t+t_m} a(s)\,ds \ge \lambda^+(a) - \varepsilon \tag{3.130}$$

for all $t_m \ge T(\varepsilon)$. Passing to the limit in (3.130) as $m \to +\infty$, we obtain the inequality

$$\frac{1}{t}\int_0^t b(s)ds \ge \lambda^+(a) - \varepsilon \tag{3.131}$$

for all $t \ge T(\varepsilon)$. From here it follows that $\mu(b) \ge \lambda^+(a) - \varepsilon$. Note that all nonzero solutions

$$\frac{dx}{dt} = b(t)x \tag{3.132}$$

are unbounded on $\mathbb{R}_+$, if $\mu(b) > 0$ and, consequently, (3.121) satisfies the condition Φ^+.

(b) $\lambda^+(a) \le \Lambda^+(a) < 0$. Then $\Lambda^+(-a) \ge \lambda^+(-a) > 0$ and, on the basis of the previous point, the equation

$$\frac{dx}{dt} = -a(t)x \tag{3.133}$$

satisfies the condition Φ^+ and, hence, (see the proof of Lemma 3.47), (3.104) also satisfies the condition Φ^+.

The second case is proved in the same way. The lemma is proved. □

From the proved theorem it follows that for L stable function a, if (3.121) satisfies the condition Φ, one of the next four cases are possible:

(1) $\Lambda^+(a) \geq \lambda^+(a) > 0$ and $\Lambda^-(a) \geq \lambda^-(a) > 0$;
(2) $\Lambda^+(a) \geq \lambda^+(a) > 0$ and $\lambda^-(a) \leq \Lambda^-(a) < 0$;
(3) $\lambda^+(a) \leq \Lambda^+(a) < 0$ and $\Lambda^-(a) \geq \lambda^-(a) > 0$;
(4) $\lambda^+(a) \leq \Lambda^+(a) < 0$ and $\lambda^-(a) \leq \Lambda^-(a) < 0$.

The following lemma gives the geometric description of each case.

Lemma 3.54. *Let a be st. L and* (3.121) *satisfy the condition Φ. Then the following conditions (a)–(d) are respectively equivalent to conditions (1)–(4).*

For every $x \neq 0$:

(a) $\lim_{t\to+\infty} |\varphi(t,a,x)| = +\infty$ *and* $\lim_{t\to-\infty} |\varphi(t,a,x)| = 0$*;*
(b) $\lim_{t\to+\infty} |\varphi(t,a,x)| = +\infty$ *and* $\lim_{t\to-\infty} |\varphi(t,a,x)| = +\infty$*;*
(c) $\lim_{t\to+\infty} |\varphi(t,a,x)| = 0$ *and* $\lim_{t\to-\infty} |\varphi(t,a,x)| = 0$*;*
(d) $\lim_{t\to+\infty} |\varphi(t,a,x)| = 0$ *and* $\lim_{t\to-\infty} |\varphi(t,a,x)| = +\infty$.

Proof. Let a be st. L, $\Lambda^+(-a) \geq \lambda^+(a) > 0$, $\lambda^-(a) \leq \Lambda^-(a) < 0$ and $x \neq 0$ $(x \in \mathbb{R})$. Let us show that $\varphi(t,a,x)$ is unbounded on $\mathbb{R}_+$. Suppose the contrary. Since (3.121) satisfies the condition Φ, then by Theorem 3.3.8 there exist positive numbers N and ν such that (3.126) holds and, consequently, $\lambda^+(a) \leq \Lambda^+(a) \leq -\nu < 0$. the last contradicts to our condition. So, $\varphi(t,a,x)$ is unbounded on $\mathbb{R}_+$ and (3.121) satisfies the condition Φ^+. According to Lemma 4.12, $\lim_{t\to+\infty} |\varphi(t,a,x)| = +\infty$. In the same way we can prove that $\lim_{t\to-\infty} |\varphi(t,a,x)| = 0$.

Conversely. Let a be st. L, (3.121) satisfy the condition Φ and for every $x \neq 0$ $(x \in \mathbb{R})$

$$\lim_{t\to+\infty} |\varphi(t,a,x)| = +\infty, \qquad \lim_{t\to-\infty} |\varphi(t,a,x)| = 0. \tag{3.134}$$

As (3.121) satisfies the condition Φ^+, then by Theorem 3.3.8, (3.121) is hyperbolic on $\mathbb{R}_+$, hence there exist positive numbers N and ν such that there takes place one of the next inequalities: (3.126) or (3.127). Note that (3.126) cannot take place as the solution $\varphi(t,a,x)$ is unbounded on $\mathbb{R}_+$. So, for the solution $\varphi(t,a,x)$ inequality (3.127) holds. It implies that

$$|\varphi(t,a,x)| \geq N^{-1}e^{\nu(t-\tau)}|\varphi(\tau,a,x)| \tag{3.135}$$

for all $t \geq \tau \geq 0$. Therefore, $\Lambda^+(a) \geq \lambda^-(a) \geq \nu > 0$. In the same way we can prove that there takes place the inequality $\Lambda^-(a) \geq \lambda^-(a) > 0$.

The other three statements are proved by the same pattern. The lemma is proved. □

Corollary 3.55. *Let a be st. L. Equation* (3.121) *satisfies the condition $\mathcal{F}$ if and only if one of conditions (1), (2) or (4) holds.*

Theorem 3.3.10. *Let a be st. L. Differential operator* (3.122) *is regular if and only if one of conditions (1) or (4) holds.*

Proof. Let a be st. L and differential operator (3.122) be regular. Then (3.121) is hyperbolic on $\mathbb{R}$, and, consequently, there exist positive numbers N and ν such that one of inequalities (3.126) and (3.127) holds. Let us consider each case separately.

Let inequality (3.126) be held. Then for $\lambda^+(a) \le \Lambda^+(a) \le -\nu < 0$ and $\lambda^-(a) \le \Lambda^-(a) \le -\nu < 0$. If (3.127) takes place, then for $t \ge \tau$

$$|\varphi(t,a,x)| \ge N^{-1}e^{\nu(t-\tau)}|\varphi(\tau,a,x)|, \tag{3.136}$$

which implies that $\Lambda^+(a) \ge \lambda^+(a) \ge \nu > 0$ and $\Lambda^-(a) \ge \lambda^-(a) \ge \nu > 0$.

Inversely. Let be fulfilled one of (1) or (4). According to Lemma 3.53, (3.121) satisfies the condition Φ. Let us consider both cases separately. Let be fulfilled condition (1). Since (3.121) satisfies the condition Φ^-, then according to Theorem 3.3.8, (3.121) is hyperbolic on $\mathbb{R}_-$. Let $f \in C_b(\mathbb{R};\mathbb{R})$. Then by Theorem 3.3.1 (see also Remark 3.52), (3.123) has at least one bounded on $\mathbb{R}_-$ solution. According to Lemma 3.54 from condition (1) it follows that all solutions of (3.121) are bounded on $\mathbb{R}_-$ and, consequently, all solutions of the nonhomogeneous (3.123) are bounded on $\mathbb{R}_-$. Theorem 3.3.1 implies that (3.123) has at least one bounded on $\mathbb{R}_+$ solution φ. By Lemma 3.54 under the condition (1) all nonzero solutions of (3.121) are unbounded on $\mathbb{R}_+$ and, consequently, φ is the single bounded on $\mathbb{R}$ solution of (3.123).

In the same way there is proved the second case. □

Theorem 3.3.11. *Let a be st. L. Differential operator* (3.122) *is weakly regular if and only if there takes place one of the following three conditions: (1), (3), or (4).*

Proof. Let a be st. L and differential operator (3.122) be weakly regular. Then it is $(C_b(I;\mathbb{R}), C_b(I;\mathbb{R}))$-admissible, where $I = \mathbb{R}_+$ or $\mathbb{R}_-$. According to Theorem 3.3.1, (3.121) is hyperbolic on $\mathbb{R}_+$ and $\mathbb{R}_-$. By Theorem 3.3.9, (3.121) satisfies the condition Φ and, hence, there takes place one of the cases (1)–(4). Let us show that (2) is impossible. In fact, assume

$$f_0(t) = \begin{cases} 0, & t \le 0, \\ t(1-t), & 0 \le t \le 1, \\ 0, & t \ge 1, \end{cases} \tag{3.137}$$

and $f(t) = f_0(t)e^{\int_0^t a(s)ds}$. Obviously, $f \in C_b(\mathbb{R};\mathbb{R})$. According to our assumption, (3.123) has at least one solution $\varphi \in C_b(\mathbb{R};\mathbb{R})$. Then there exists $x_0 \in \mathbb{R}$ such that

$$\varphi(t) = \left(x_0 + \int_0^t f_0(s)ds\right)e^{\int_0^t a(s)ds}. \tag{3.138}$$

From equalities (3.137) and (3.138) it follows that

$$\varphi(t) = \begin{cases} e^{\int_0^t a(s)ds} x_0, & t \leq 0, \\ e^{\int_0^t a(s)ds}\left(x_0 + \dfrac{1}{6}\right), & t \geq 1. \end{cases} \tag{3.139}$$

From (3.139) it follows that for $t \leq 0$ and $t \geq 1$ φ is the solution of the homogeneous (3.121) and, consequently, all nonzero solutions of (3.123) are bounded at least one of the semiaxis $\mathbb{R}_+$ or $\mathbb{R}_-$. So, (2) is impossible.

Conversely. Let one of the conditions (1), (3), or (4) be fulfilled. If there holds (1) or (4), then by Theorem 3.3.10 differential operator (3.122) is regular and the theorem is proved. Let now (3) be fulfilled. According to Lemma 3.53, (3.121) satisfies the condition Φ and from Lemma 3.54 it follows that all nonzero solutions of (3.121) are bounded on $\mathbb{R}$. Since (3.121) satisfies the condition Φ, then by Theorem 3.3.9, (3.121) is hyperbolic on $\mathbb{R}_+$ and $\mathbb{R}_-$. Let $f \in C_b(\mathbb{R};\mathbb{R})$. By Theorem 3.3.1, (3.123) has at least two solutions one of which is bounded on $\mathbb{R}_+$ and other is bounded on $\mathbb{R}_-$. Therefore, all solutions of (3.123) are bounded on $\mathbb{R}$. The theorem is proved. □

Corollary 3.56. *Let a be st. L. Differential operator* (3.122) *is weakly regular, if and only if* (3.123) *satisfies the condition* $\mathcal{F}$.

Corollary 3.57. *Let a be st. L and* (3.121) *satisfy the condition* $\mathcal{F}$. *Then either differential operator* (3.122) *or*

$$L_{(-a)}x = \frac{dx}{dt} + a(t)x \tag{3.140}$$

is weakly regular.

Corollary 3.58. *Let a be st. L. Equation* (3.121) *is hyperbolic on* $\mathbb{R}$, *if and only if equations* (3.121) *and* (3.133) *satisfy the condition* $\mathcal{F}$.

Corollary 3.59. *Let a be st. L. Equation* (3.121) *satisfies the condition* $\mathcal{F}$, *if and only if differential operator* (3.122) *is weakly regular.*

3.3.4.2. Triangular Systems

In this section we consider the matrix $A = (a_{ij})$ upper triangular ($a_{ij} = 0$ for $i > j$).

Lemma 3.60. *Let A be st.* L^+ *(resp., st.* L^-*). Equation* (3.121) *satisfies the condition* Φ^+ *(resp.,* Φ^-*), if and only if for every* $i = 1, 2, \ldots, n$ *the differential equation*

$$\frac{dx}{dt} = a_{ii}(t)x \tag{3.141}$$

satisfies the condition Φ^+ *(resp.,* Φ^-*).*

Proof. Let for every $i = 1, 2, \ldots, n$ (3.141) satisfies the condition Φ^+. Let us show that (3.121) satisfies the condition Φ^+ too. Suppose the contrary. Then there exists $B \in \omega_A$ such that

$$\frac{dy}{dt} = B(t)y \tag{3.142}$$

has nonzero solution $\varphi \in C_b(\mathbb{R}; E^n)$. Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, φ_{k_0} $(1 \le k_0 \le n)$ is the last nonzero component. Then $\varphi_{k_0} \in C_b(\mathbb{R}; \mathbb{R})$ and

$$\frac{d\varphi_{k_0}(t)}{dt} = b_{k_0k_0}(t)\varphi_{k_0}(t). \tag{3.143}$$

Since $B \in \omega_A$, there exists $t_m \to +\infty$ such that $B := \lim_{m\to+\infty} A^{(t_m)}$. Obviously,

$$b_{k_0k_0} = \lim_{m\to+\infty} a_{k_0k_0}^{(t_m)}, \tag{3.144}$$

consequently, $b_{k_0k_0} \in \omega_{a_{k_0k_0}}$. So, we found $1 \le k_0 \le n$ such that (3.141) does not satisfy the condition Φ^+ for $i = k_0$. The last contradicts to our condition.

Inversely. Let A be st. L^+ and (3.121) satisfy the condition Φ^+. Let us show that for every $i = 1, 2, \ldots, n$ (3.141) satisfies the condition Φ^+. The proof will be carried out by induction by the dimensionality n of the system. For $n = 1$ the statement is obvious. Suppose that the lemma is true for all $n \le k - 1$. Let us show that it is true also for $n = k$. The fact that (3.121) with $A \in C(\mathbb{R}; [\mathbb{R}^k])$ satisfies the condition Φ^+ for $i = 1$ implies that (3.141) also satisfies the condition Φ^+ for $i = 1$. In fact, if we suppose the contrary, then there exists $b_{11} \in \omega_{a_{11}}$ such that the equation

$$\frac{dx}{dt} = b_{11}(t)x \tag{3.145}$$

has a nonzero solution $\varphi_1 \in C_b(\mathbb{R}; \mathbb{R})$. As $b_{11} \in \omega_{a_{11}}$, there exists $t_m \to +\infty$ such that $b_{11} = \lim_{m\to+\infty} a_{11}^{(t_m)}$. In virtue of the L^+ stability of the matrix A, the sequence $\{A^{(t_m)}\}$ can be considered convergent. Put $B := \lim_{m\to+\infty} A^{(t_m)}$. Then the function $\varphi = (\varphi_1, 0, 0, \ldots, 0) \in C_b(\mathbb{R}; \mathbb{R}^k)$ is a nonzero solution of (3.142). The last contradicts to the condition. So, (3.141) satisfies the condition Φ^+ for $i = 1$. Let us show that (3.121) with the matrix

$$\widetilde{A} = (a_{ik})_{i,j=2}^{k} \tag{3.146}$$

satisfies the condition Φ^+. Suppose the contrary. Then there exists $\widetilde{B} \in \omega_{\widetilde{A}}$ such that

$$\frac{dx}{dt} = \widetilde{B}(t)x \tag{3.147}$$

has a nonzero solution $\widetilde{\varphi} = (\varphi_2, \varphi_3, \ldots, \varphi_k) \in C_b(\mathbb{R}; \mathbb{R}^{k-1})$. As $A \in C(\mathbb{R}; [\mathbb{R}^k])$ is st. L^+ and $\widetilde{B} \in \omega_{\widetilde{A}}$, then there exists $t_m \to +\infty$ such that

$$\widetilde{B} := \lim_{m\to+\infty} \widetilde{A}^{(t_m)}, \qquad B := \lim_{m\to+\infty} A^{(t_m)}, \tag{3.148}$$

and $\widetilde{b}_{ij} = b_{ij}$ $(i, j = 2, 3, \ldots, k)$. Consider the equation

$$\frac{dx}{dt} = b_{11}(t)x + b_{12}(t)\varphi_2(t) + \cdots + b_{1k}\varphi_k(t). \tag{3.149}$$

Since (3.141) for $i = 1$ satisfies the condition Φ^+, then from Theorems 3.3.9, 3.3.8, and Lemma 3.32 it follows that (3.149) has a bounded on $\mathbb{R}$ solution φ_1. It is easy to see that the function $(\varphi_1, \varphi_2, \dots, \varphi_k) \in C_b(\mathbb{R}; \mathbb{R}^k)$ is a nonzero solution of (3.142), where $B \in \omega_A$. The last contradicts to the condition of the lemma. So, differential equation (3.121) with the matrix (3.146) satisfies the condition Φ^+. In virtue of the inductive assumption for every $i = 2, 3, \dots, n$ (3.141) satisfies the condition Φ^+. In the same way the second case can be considered. The lemma is proved. □

Corollary 3.61. *Let $A \in C(\mathbb{R}; [\mathbb{R}^n])$ be st. L. Equation* (3.121) *satisfies the condition Φ, if and only if for every $i = 1, 2, \dots, n$* (3.141) *satisfies the condition Φ.*

Lemma 3.62. *Let $A \in C(\mathbb{R}; [\mathbb{R}^n])$ be such that* (3.121) *is $(C_b(\mathbb{R}; \mathbb{R}^n), C_b(\mathbb{R}; \mathbb{R}^n))$ admissible and $B \in C(\mathbb{R}; [\mathbb{R}^m])$ be such that* (3.142) *is $(C_b(\mathbb{R}; \mathbb{R}^m), C_b(\mathbb{R}; \mathbb{R}^m))$ admissible. Then equation*

$$\frac{dx}{dt} = C(t)x, \tag{3.150}$$

where

$$C = \begin{pmatrix} A & 0 \\ C' & B \end{pmatrix} \tag{3.151}$$

is $(C_b(\mathbb{R}; \mathbb{R}^{n+m}), C_b(\mathbb{R}; \mathbb{R}^{n+m}))$ admissible.

The proof is obvious.

3.3.4.3. Systems of General Form

Lemma 3.63. *Let $\varphi \in C(\mathbb{R}; E^n)$. The following conditions are equivalent:*

(a) $\lim_{t\to+\infty} |\varphi(t)| = 0$ *(resp.,* $\lim_{t\to-\infty} |\varphi(t)| = 0$*);*

(b) *φ is st. L^+ (resp., L^-) and $\omega_\varphi = \{\theta\}$ (resp., $\alpha_\varphi = \{\theta\}$), where $\theta \in C(\mathbb{R}; E^n)$ is a function that is identically equal to zero.*

Proof. Let $\lim_{t\to+\infty} |\varphi(t)| = 0$. Then $\overline{\varphi(\mathbb{R}_+)}$ is a compact set in E^n. Let us show that φ is uniformly continuous on $\mathbb{R}_+$. Suppose the contrary. Then there exist $\varepsilon_0 > 0$, $\delta_m \to 0$ and $t_m^{(i)} \to +\infty$ $(i = 1, 2)$ such that

$$|t_m^{(1)} - t_m^{(2)}| < \delta_m, \qquad |\varphi(t_m^{(1)}) - \varphi(t_m^{(2)})| \geq \varepsilon_0. \tag{3.152}$$

Passing to the limit in (3.152) we get $\varepsilon_0 \leq 0$. The last contradicts to the choice of the number ε_0. So, φ has compact values on $\mathbb{R}_+$ (i.e., the set $\varphi(\mathbb{R}_+)$ is a relatively compact set) and, consequently [92], it is st. L^+. Let $\psi \in \omega_\varphi$ then there exists $t_m \to +\infty$ such that $\psi(t) = \lim_{m\to+\infty} \varphi(t + t_m)$ for every $t \in \mathbb{R}$ and, hence, $\psi \equiv 0$.

Inversely. Let φ be st. L^+ and $\omega_\varphi = \{\theta\}$. Let us show that $\lim_{t\to+\infty} |\varphi(t)| = 0$. Suppose the contrary. Then there exists $\varepsilon_0 > 0$ and $t_m \to +\infty$ such that

$$|\varphi(t_m)| \geq \varepsilon_0. \tag{3.153}$$

In virtue of the L^+ stability of the function φ, the sequence $\{\varphi^{(t_m)}\}$ can be considered convergent on $C(\mathbb{R};E^n)$. Assume $\psi := \lim_{m\to+\infty}\varphi^{(t_m)}$. From inequality (3.153) it follows that $|\psi(0)| \geq \varepsilon_0$. Besides, $\psi \in \omega_\varphi = \{\theta\}$ and, consequently, $\psi \equiv 0$.

The last contradicts to the choice of the number ε_0. The lemma is proved. □

Consider the differential operator

$$L_A^* x = \frac{dx}{dt} + A_*(t)x, \tag{3.154}$$

that is formally adjoint to operator (3.122), where $A_*(t)$ is the matrix adjoint to $A(t)$.

Lemma 3.64. *Let L_A and $L_A^* : C_b^1(\mathbb{R};E^n) \to C_b(\mathbb{R};E^n)$ and $A \in C_b(\mathbb{R};[E^n])$. Then*

$$\operatorname{Ker} L_A^* \cap \operatorname{Im} L_A = \{0\}, \tag{3.155}$$

where $\operatorname{Ker} L_A^*$ is the kernel of the operator L_A^*, and $\operatorname{Im} L_A$ is the domain of values of the operator L_A.

Proof. Let $\varphi \in \operatorname{Ker} L_A^* \cap \operatorname{Im} L_A$. Then $L_A^*\varphi = 0$ and there exists $\psi \in C_b^1(\mathbb{R};E^n)$ such that $L_A\psi = \varphi$. Consider the function $\gamma \in C_b(\mathbb{R};\mathbb{R})$ defined by the equality

$$\gamma(t) := \langle \varphi(t), \psi(t)\rangle, \quad (t \in \mathbb{R}), \tag{3.156}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in E^n. Then

$$\dot\gamma(t) = |\varphi(t)|^2 \quad (t \in \mathbb{R}). \tag{3.157}$$

From equality (3.157) it follows that there exist limits

$$\lim_{t\to+\infty}\gamma(t) = \alpha, \qquad \lim_{t\to-\infty}\gamma(t) = \beta. \tag{3.158}$$

Let us show that $\alpha = \beta = 0$. Since $A \in C_b(\mathbb{R};[E^n])$, then φ is uniformly continuous on $\mathbb{R}$ and, consequently [92], it is st. L, $\omega_\varphi \neq \emptyset$, and $\alpha_\varphi \neq \emptyset$. Let $\widetilde\varphi \in \omega_\varphi$. Then there exists $t_m \to +\infty$ such that $\widetilde\varphi = \lim_{m\to+\infty}\varphi^{t_m}$. Note that ψ is st. L, therefore γ is also st. L and, consequently, $\{\gamma^{t_m}\}$ can be considered convergent on $C(\mathbb{R};\mathbb{R})$. Put $\widetilde\gamma := \lim_{m\to+\infty}\gamma^{t_m}$. From (3.158) it follows that $\widetilde\gamma(t) = \alpha$ for all $t \in \mathbb{R}$. Passing to the limit in the equality

$$\gamma'(t+t_m) = |\varphi(t+t_m)|^2 \tag{3.159}$$

and taking into consideration the above said, we obtain $\widetilde\gamma'(t) = |\widetilde\varphi(t)|^2$. Hence $\widetilde\varphi(t) = 0$ for all $t \in \mathbb{R}$. So, φ is st. L and $\omega_\varphi = \{\theta\}$. According to Lemma 3.63, $\lim_{t\to+\infty}|\varphi(t)| = 0$. In the same way we prove that $\lim_{t\to-\infty}|\varphi(t)| = 0$. Note that

$$|\gamma(t)| = |\langle\varphi(t),\psi(t)\rangle| \leq |\psi(t)|\,|\varphi(t)|. \tag{3.160}$$

Since $\psi \in C_b(\mathbb{R};E^n)$, then

$$\lim_{|t|\to+\infty}\gamma(t) = 0. \tag{3.161}$$

But the nondecreasing function $\gamma \in C(\mathbb{R};\mathbb{R})$ satisfies condition (3.161) if and only if $\gamma \equiv 0$. From (3.157) it follows that $\varphi \equiv 0$. □

Corollary 3.65. *Let* $A \in C_b(\mathbb{R};[E^n])$ *and* $\operatorname{Im} L_A = C_b(\mathbb{R};E^n)$. *Then* $\operatorname{Ker} L_A^* = \{\theta\}$.

Lemma 3.66. *Let A be st. L and* $\operatorname{Im} L_A = C_b(\mathbb{R};E^n)$. *Then for every* $B \in H(A)$ *there takes place the equality* $\operatorname{Im} L_B = C_b(\mathbb{R};E^n)$.

Proof. Let $\operatorname{Im} L_A = C_b(\mathbb{R};E^n)$ and $B \in H(A)$. Logically, two cases are possible:

(1) there exists $\tau \in \mathbb{R}$ such that $B = A^{(\tau)}$. In this case the lemma is obvious;

(2) $B \in \triangle_A$. According to Theorem 3.3.9, (3.121) satisfies the condition Φ and from Theorem 3.3.8 and Lemma 3.32 it follows that (3.142) satisfies the condition of exponentially dichotomy on $\mathbb{R}$. The lemma is proved. □

Corollary 3.67. *Let A be st. L and* $\operatorname{Im} L_A = C_b(\mathbb{R};E^n)$. *Then for every* $B \in H(A)$ *there takes place the equality* $\operatorname{Ker} L_B^* = \{\theta\}$, *that is, the equation*

$$\frac{dx}{dt} = -A_*(t)x \tag{3.162}$$

satisfies the condition Φ.

Lemma 3.68. *Equation* (3.121) *is hyperbolic on* $\mathbb{R}_+$ *(resp.,* $\mathbb{R}_-, \mathbb{R}$*), if and only if* (3.162) *is hyperbolic* $\mathbb{R}_+$ *(resp.,* $\mathbb{R}_-, \mathbb{R}$*).*

Corollary 3.69. *Let A be st.* L^+ *(resp.,* L^-*). Equation* (3.121) *satisfies the condition* Φ^+ *(resp.,* Φ^-*), if and only if* (3.121) *satisfies the condition* Φ^+ *(resp.,* Φ^-*).*

The formulated statement follows from Theorem 3.3.9 and Lemma 3.68.

Corollary 3.70. *Let A be st. L. Equation* (3.121) *satisfies the condition* Φ, *if and only if* (3.162) *also does.*

Lemma 3.71 (see [124]). *If* (3.121) *with the help of Lyapunov's transformation* $x(t) = L(t)y(t)$ *can be reduced to* (3.142), *then* (3.162) *can be reduced to the equation*

$$\frac{dy}{dt} = -B_*(t)y \tag{3.163}$$

with the help of Lyapunov's transformation $x(t) = L_*^{-1}(t)y(t)$.

Corollary 3.72. *If with the help of Lyapunov's unitary transformation* $x(t) = L(t)y(t)$ (3.121) *can be reduced to* (3.142), *then with the help of the same transformation* (3.162) *can be reduced to* (3.163).

Lemma 3.73. *Let* (3.121) *be hyperbolic on* $\mathbb{R}_+$ *and all nonzero solutions of* (3.121) *are unbounded on* $\mathbb{R}_+$ *(resp.,* $\mathbb{R}_-$*). Then there exists positive numbers* N *and* ν *such that*

$$||U(t,-A_*)|| \le Ne^{-\nu|t|} \tag{3.164}$$

for all $t \in \mathbb{R}_+$ *(resp.,* $\mathbb{R}_-$*).*

Proof. Let (3.121) be hyperbolic on $\mathbb{R}_+$ and all nonzero solutions of (3.121) be unbounded on $\mathbb{R}_+$. Then there exist positive projectors P and Q ($P+Q=E$) and positive numbers N and ν such that

$$||U(t,A)PU^{-1}(\tau,A)|| \le Ne^{-\nu(t-\tau)} \quad (t \ge \tau \ge 0), \tag{3.165}$$

$$||U(t,A)QU^{-1}(\tau,A)|| \le Ne^{\nu(t-\tau)} \quad (0 \le t \le \tau). \tag{3.166}$$

Since all nonzero solutions of (3.121) are unbounded on $\mathbb{R}_+$, then $P = 0$ and, consequently, $\|U(t,A)U^{-1}(\tau,A)\| \le Ne^{-\nu(t-\tau)}$ $(t \ge \tau \ge 0)$. In particular, for all $t \in \mathbb{R}_+$ there takes place inequality $\|U^{-1}(t,A)\| \le Ne^{-\nu t}$. Let $U(t,-A_*)$ be the Cauchy operator of (3.162). Then [116] $U(t,-A_*) = U_*^{-1}(t,A)$. Therefore, inequality (3.164) holds for all $t \in \mathbb{R}_+$.

In the same way we consider the second case. The lemma is proved. □

Corollary 3.74. *Let* (3.121) *be hyperbolic on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$*. If all nonzero solutions of* (3.121) *are unbounded on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$*, then there exist positive numbers* N *and* ν *such that for all* $t \in \mathbb{R}$ *there is fulfilled inequality* (3.164).

Corollary 3.75. *Let* A *be st.* L^+ *(resp.,* L^-, L*) and* (3.121) *satisfy the condition* Φ^+ *(resp.,* Φ^-, Φ*). If all nonzero solutions of* (3.121) *are unbounded on* $\mathbb{R}_+$ *(resp.,* $\mathbb{R}_-, \mathbb{R}_+$*, and* $\mathbb{R}_-$*), then there exist positive numbers* N *and* ν *such that inequality* (3.164) *holds for all* $t \in \mathbb{R}_+$ *(resp.,* $\mathbb{R}_-, \mathbb{R}$*).*

Lemma 3.76. *Let* A *be st.* L *and satisfy the condition* Φ*. If all solutions of* (3.121) *are bounded on* $\mathbb{R}_+$ *(resp.,* $\mathbb{R}_-, \mathbb{R}$*), then the operator* L_A *is weakly regular.*

Proof. Let A be st. L, (3.121) satisfy the condition Φ, all nonzero solutions of (3.142) be bounded on $\mathbb{R}_+$, and $f \in C_b(\mathbb{R};E^n)$. Since (3.121) satisfies the condition Φ^+, (3.123) has at least one solution from $C_b(\mathbb{R}_+,E^n)$ and, consequently, all solutions of (3.123) are bounded on $\mathbb{R}_+$. Besides, (3.121) satisfies the condition Φ^- and therefore (3.123) has at least one bounded on $\mathbb{R}_-$ solution φ. Obviously, $\varphi \in C_b(\mathbb{R};E^n)$. In the same way we can consider the other two cases. The lemma is proved. □

Corollary 3.77. *Let* A *be st.* L*,* (3.121) *satisfy the condition* Φ *and every nonzero solution of* (3.162) *be unbounded on* $\mathbb{R}_+$ *(resp.,* $\mathbb{R}_-$*,* $\mathbb{R}_+$*, and* $\mathbb{R}_-$*). Then operator* L_A *is weakly regular.*

This statement follows from Lemma 3.76 and Corollary 3.75.

Lemma 3.78. *Let A be st. L and* (3.121) *satisfy the condition Φ. If* (3.121) *has nonzero solution φ_0 bounded on $\mathbb{S}$ ($\mathbb{S} = \mathbb{R}_+$, $\mathbb{R}_-$ or $\mathbb{R}$), then there exists unitary Lyapunov's transformation $x(t) = L(t)y(t)$ that brings* (3.121) *to the triangular form* (3.142) *with the triangular matrix*

$$B(t) = \left(b_{ij}(t)\right)_{i,j=1}^{n}, \tag{3.167}$$

and in this case (3.145) *is $(C_b(\mathbb{R};\mathbb{R}), C_b(\mathbb{R};\mathbb{R}))$-admissible, if $\varphi_0 \in C_b(\mathbb{R};\mathbb{R}^n)$, and $(C_b(\mathbb{R};\mathbb{R}), C_b(\mathbb{R};\mathbb{R}))$-regular otherwise.*

Proof. Let A be st. L, (3.121) satisfy the condition Φ and φ_0 be a nonzero bounded on $\mathbb{S}$ solution of (3.121). Let $x^{(1)}, x^{(2)}, \dots, x^{(n)}$ be a base of the space of solutions of (3.121) and $x^{(1)} = \varphi_0$. Applying to this basis the theorem of Perrone [124], we get the first statement of the lemma. From the same theorem (see also [116]) it follows that

$$b_{11}(t) = \frac{d}{dt} \ln \left|\varphi_0(t)\right|. \tag{3.168}$$

To finish the proof of the lemma it is enough to apply Lemma 3.54. □

Theorem 3.3.12. *Let A be st. L. The differential operator L_A is weakly regular, if and only if* (3.162) *satisfies the condition $\mathcal{F}$.*

Proof. Let A be st. L and L_A be weakly regular. According to Corollary 3.67, (3.162) satisfies the condition $\mathcal{F}$.

Inversely. Let (3.162) satisfy the condition $\mathcal{F}$. Let us show that the differential operator L_A is weakly regular. We will prove it by the induction by the dimensionality n of the system. For $n = 1$ the validity of the theorem follows from Corollary 3.56. Suppose that it is true for all $n \le k - 1$. Show now that it is true for $n = k$ too. Logically, two cases are possible.

(1) All nonzero solutions of (3.162) are unbounded on $\mathbb{R}_+$. According to Corollary 3.77, the differential operator L_A is weakly regular and the lemma is proved.

(2) There exists a nonzero bounded on $\mathbb{R}_+$ (and, consequently, unbounded on $\mathbb{R}_-$) solution φ_0. By Lemma 3.78, with the help of the unitary Lyapunov's transformation (3.162) can be reduced to the triangular form

$$\begin{aligned} \dot{v}_1 &= b_{11}(t)v_1 - b_{12}(t)v_2 - \cdots - b_{1n}(t)v_n, \\ \dot{v}_2 &= \qquad\qquad - b_{22}(t)v_2 - \cdots - b_{2n}(t)v_n, \\ &\qquad\qquad\qquad \cdots \\ \dot{v}_n &= \qquad\qquad\qquad\qquad\qquad - b_{nn}(t)v_n, \end{aligned} \tag{3.169}$$

and equation

$$\dot{u}_1 = b_{11}(t)u_1 \tag{3.170}$$

is $(C_b(\mathbb{R};\mathbb{R}), C_b(\mathbb{R};\mathbb{R}))$-regular. Consider the matrix

$$-\widetilde{B}(t) = \left(-b_{ij}\right)_{i,j=2}^{n}. \tag{3.171}$$

Let us show that the equation

$$\dot{x} = -\widetilde{B}(t)x \tag{3.172}$$

satisfies the condition $\mathcal{F}$. Under the condition of the theorem (3.162) satisfies the condition $\mathcal{F}$ and, hence, (3.169) satisfies it too. Corollary 3.61 implies that (3.172) satisfies the condition Φ. Suppose that (3.172) does not satisfy the condition $\mathcal{F}$. Then there exists a nonzero solution $(\varphi_2, \varphi_3, \dots, \varphi_n) \in C_b(\mathbb{R}; \mathbb{R}^{n-1})$ of (3.172). Since (3.170) is $(C_b(\mathbb{R}; \mathbb{R}), C_b(\mathbb{R}; \mathbb{R}))$ regular, (3.169) has nonzero from $C_b(\mathbb{R}; \mathbb{R}^n)$ too. The last contradicts to the condition of the theorem. So, (3.172) satisfies the condition $\mathcal{F}$ and, in virtue of the inductive assumption the equation

$$\dot{y} = \widetilde{B}_*(t)y \tag{3.173}$$

is $(C_b(\mathbb{R}; \mathbb{R}^{n-1}), C_b(\mathbb{R}; \mathbb{R}^{n-1}))$ admissible. Since (3.170) also is $(C_b(\mathbb{R}; \mathbb{R}), C_b(\mathbb{R}; \mathbb{R}))$ regular, (3.170) is $(C_b(\mathbb{R}; \mathbb{R}^{n-1}), C_b(\mathbb{R}; \mathbb{R}^{n-1}))$ regular too. By Lemma 3.62, equation

$$\dot{y} = B_*(t)y, \tag{3.174}$$

where

$$\begin{pmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ b_{12} & b_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{1n} & b_{2n} & b_{3n} & \cdots & b_{nn} \end{pmatrix} \tag{3.175}$$

is $(C_b(\mathbb{R}; \mathbb{R}^n), C_b(\mathbb{R}; \mathbb{R}^n))$ admissible. From Lemma 3.71 it follows that the operator L_A is weakly regular. The theorem is proved. □

Corollary 3.79. *Let A be st. L. The differential operator L_A is regular, if and only if* (3.121) *and* (3.162) *satisfy the condition $\mathcal{F}$.*

Corollary 3.80. *Let A be st. L. Equation* (3.121) *satisfies the condition $\mathcal{F}$, if and only if the operator L_A^* is weakly regular.*

3.3.5. Correct Differential Operators

Let $A \in C(\mathbb{R}; [E^n])$. Consider the differential operator

$$L_A x = \frac{dx}{dt} - A(t)x \tag{3.176}$$

in the space $C_b(\mathbb{R}; E^n)$, considering that it is defined on such functions as $x \in C_b(\mathbb{R}; E^n)$ for which $L_A x \in C_b(\mathbb{R}; E^n)$.

Definition 3.81. The operator L_A is called correct, if there exists a positive number $\delta > 0$ such that the estimation

$$||L_A u|| \geq \delta \|u\| \tag{3.177}$$

is true for all u, $L_A u \in C_b(\mathbb{R}; E^n)$.

Definition 3.82. The operator L_A is called uniformly correct, if there exists a positive number $\delta > 0$ such that

$$||L_B u|| \geq \delta \|u\| \tag{3.178}$$

for every $B \in H(A)$ and u, $L_B u \in C_b(\mathbb{R}; E^n)$.

Obviously, any uniformly correct operator L_A is correct. The inverse statement apparently is not true, though the respective example is unknown for us.

There takes place.

Theorem 3.3.13. *Let the operator-function $A \in C(\mathbb{R}; [E^n])$ be almost periodic and the operator L_A be correct. Then L_A is uniformly correct too.*

Proof. Let $A \in C(\mathbb{R}; [E^n])$ be almost periodic and $B \in H(A)$. Since the operator L_A is correct, then there exists a positive number $\delta > 0$ such that inequality (3.177) holds. For the number $\delta/2$ in virtue of the almost periodicity of A there exists a number $\tau \in \mathbb{R}$ such that

$$||A(t+\tau) - B(t)|| < \varepsilon \quad (t \in R). \tag{3.179}$$

Let u, $L_B u \in C_b(\mathbb{R}; E^n)$. Then

$$\begin{aligned} ||L_B u|| &= ||L_{[A^{(\tau)}+B-A^{(\tau)}]} u|| \\ &\geq ||L_{A^{(\tau)}} u|| - \sup_{t\in\mathbb{R}} \left| [B(t) - A(t+\tau)]u(t) \right| \geq \delta\|u\| - \frac{\delta}{2}\|u\| = \frac{\delta}{2}\|u\|. \end{aligned} \tag{3.180}$$

The theorem is proved. □

Lemma 3.83. *Let $A \in C(\mathbb{R}; [E^n])$ and operator L_A be correct. Then there exists a number $\gamma > 0$ such that L_B is correct, if*

$$||B(t) - A(t)|| < \gamma \quad (t \in \mathbb{R}). \tag{3.181}$$

Proof. The formulated statement is proved in the same way that Theorem 3.3.13. □

Corollary 3.84. *The set of correct operators L_A forms an open set (by the operator norm) in the set of all differential operators* (3.176).

Corollary 3.85. *The set of uniformly correct operators L_A with the almost periodic function $A(t)$ is open in the set of all almost periodic differential operators.*

Theorem 3.3.14. *Let $A \in C(\mathbb{R}; [E^n])$ be st. L. The next conditions are equivalent:*

(1) *the operator L_A is uniformly correct;*

(2) *the differential equation*

$$\frac{dx}{dt} = A(t)x \tag{3.182}$$

satisfies the condition $\mathcal{F}$.

Proof. Let L_A be uniformly correct. Then there exists $\delta > 0$ such that for every $\in H(A)$ there is fulfilled inequality (3.178). Let us show that (3.182) satisfies the condition $\mathcal{F}$. Suppose the contrary. Then there exists $B \in H(A)$ and nonzero function $\varphi \in C_b(\mathbb{R}; E^n)$ such that

$$L_B\varphi = 0. \tag{3.183}$$

From (3.178) and (3.183) it follows that $\delta \leq 0$. The last contradicts to the choice of the number δ.

Inversely. Let A be st. L and (3.182) satisfy the condition $\mathcal{F}$. Let us show that the operator L_A is uniformly correct. Suppose the contrary, that is, there exist $\{\varphi_m\} \subseteq C_b(\mathbb{R}; E^n)$, $\{B_m\} \subseteq H(A)$, and $\alpha_m \to 0$ ($\alpha_m > 0$) such that

$$||\varphi_m|| = 1, \qquad ||L_{B_m}\varphi_m|| \leq \alpha_m \tag{3.184}$$

for all $m \in \mathbb{N}$. Condition (3.184) implies the existence of the sequence $\{t_m\} \subseteq \mathbb{R}$ such that

$$|\varphi_m(t_m)| \geq \frac{1}{2}, \tag{3.185}$$

$$\left| \frac{d\varphi_m(t)}{dt} - B_m(t)\varphi_m(t) \right| \leq \alpha_m \tag{3.186}$$

for all $t \in \mathbb{R}$. Define the sequence $\{\psi_m\}$ by the equality

$$\psi_m(t) = \varphi_m(t + t_m) \quad (t \in R). \tag{3.187}$$

From (3.186) it follows that

$$\left| \frac{d\psi_m(t)}{dt} - B_m(t + t_m)\psi_m(t) \right| \leq \alpha_m \tag{3.188}$$

for all $t \in \mathbb{R}$. Let us show that $\{\psi_m\}$ is relatively compact in $C(\mathbb{R}; E^n)$. In fact, it is easy to see that for every $m \in \mathbb{N}$ the function ψ_m is a solution of the differential equation

$$\frac{dx}{dt} = B_m(t + t_m)x + g_m(t), \tag{3.189}$$

where $g_m(t) := \psi'_m(t) - B_m(t + t_m)\psi_m(t)$ for all $t \in \mathbb{R}$. From (3.188) it follows that $g_m \to 0$ in the topology $C(\mathbb{R}; E^n)$. In virtue of the L stability of the operator-function $A(t)$, the sequence $\{B_m^{(t_m)}\}$ can be considered convergent in $C(\mathbb{R}; [E^n])$. Note that $\|\psi_m\| \leq 1$

and by Lemma 3.37 the sequence $\{\psi_m\}$ is relatively compact in $C(\mathbb{R}; E^n)$. Without loss of generality the sequences $\{\psi_m\}$ and $\{B_m^{(t_m)}\}$ can be considered convergent. Assume $\psi_0 := \lim_{m\to+\infty}\psi_m$ and $B_0 := \lim_{m\to+\infty} B_m^{(t_m)}$. By Lemma 3.37 ψ_0 is a bounded on $\mathbb{R}$ solution of the equation

$$\frac{du}{dt} = B_0(t)u. \tag{3.190}$$

From inequality (3.185) it follows that $\psi_0 \not\equiv 0$. The last contradicts to the condition of the theorem. The theorem is proved. □

Theorem 3.3.15. *Let $A \in C(\mathbb{R}; [E^n])$ be st. L and* (3.182) *satisfy the condition $\mathcal{F}$. Then there exists a positive number $\gamma > 0$ such that the equation*

$$\dot{x} = B(t)x \tag{3.191}$$

also satisfies the condition $\mathcal{F}$, if

$$||B(t) - A(t)|| < \gamma \quad (t \in \mathbb{R}), \tag{3.192}$$

where $B \in C(\mathbb{R}; [E^n])$.

Proof. Let $A \in C(\mathbb{R}; [E^n])$ be st. L and (3.182) satisfy the condition $\mathcal{F}$. According to Theorem 3.3.14, there exists a positive number $\delta > 0$ such that inequality (3.178) holds for every $B \in H(A)$. Assume $\gamma = \delta/2$. Let $B \in C(\mathbb{R}; [E^n])$ be such that inequality (3.192) be fulfilled. Then for every $\varphi \in C_b(\mathbb{R}; E^n)$ from the domain of definition of L_B

$$||L_B\varphi|| = ||L_B\varphi - (B - A)\varphi|| \geq ||L_A\varphi|| - ||(B - A)\varphi|| \geq \delta\|\varphi\| - \gamma\|\varphi\| = \left(\frac{\delta}{2}\right)\|\varphi\|. \tag{3.193}$$

Let $C \in H(B)$. Then there exists $\{t_m\} \subseteq \mathbb{R}$ such that $C = \lim_{m\to+\infty} B^{(t_m)}$ in $C(\mathbb{R}; [E^n])$. Note that for each $m \in \mathbb{N}$ and for all $\varphi \in C_b(\mathbb{R}; E^n)$

$$||L_{B^{(t_m)}}\varphi|| = ||L_B\varphi^{(-t_m)}|| \geq \left(\frac{\delta}{2}\right)||\varphi^{(-t_m)}|| = \left(\frac{\delta}{2}\right)\|\varphi\|. \tag{3.194}$$

The last inequality can be rewritten in the form

$$\left|\frac{d\varphi(t)}{dt} - B(t + t_m)\varphi(t)\right| \geq \left(\frac{\delta}{2}\right)\|\varphi\| \quad (t \in \mathbb{R}). \tag{3.195}$$

Passing to the limit in inequality (3.195) as $m \to +\infty$, we get

$$||L_C\varphi|| \geq \left(\frac{\delta}{2}\right)\|\varphi\| \tag{3.196}$$

for all $C \in H(A)$. From inequality (3.196) and Theorem 3.3.14 the necessary statement follows. The theorem is proved. □

Corollary 3.86. *The set of all* (3.182) *with the st. L operator-function A satisfying the condition $\mathcal{F}$ forms an open set (in the uniform topology) in the space of all equations of form* (3.182).

Theorem 3.3.16. *Let $A \in C(\mathbb{R}; [E^n])$ be st. L. The operator L_A is weakly regular, if and only if the operator*

$$L_A^* x = \frac{dx}{dt} + A_*(t)x \tag{3.197}$$

is uniformly correct.

Proof. The formulated statement is the sequence of Theorems 3.3.12 and 3.3.14. □

So, Theorem 3.3.16 establishes the duality of the notions of uniform correctness and weak regularity.

3.3.6. Linear Equations with Asymptotically Almost Periodic Coefficients

Let $C(\mathbb{R}, [E^n])$ be the space of all the continuous matrix-functions $A : \mathbb{R} \to [E^n]$ with the compact-open topology.

Let us consider a differential equation

$$\frac{dx}{dt} = A(t)x, \tag{3.198}$$

where $A \in C(\mathbb{R}, [E^n])$. Along with (3.198) we consider a nonhomogeneous equation

$$\frac{dy}{dt} = A(t)y + f(t), \tag{3.199}$$

where $f \in C(\mathbb{S}, E^n)$, and the family of "ω-limit" equations

$$\frac{dz}{dt} = B(t)z \quad (B \in \omega_A). \tag{3.200}$$

Theorem 3.3.17. *Let φ be a bounded on $\mathbb{R}_+$ solution of* (3.199), *the matrix $A \in C(\mathbb{R}, [E^n])$ and the function f be st. L^+. If every equation of family* (3.200) *has no nontrivial bounded on $\mathbb{R}$ solutions, then φ is compatible in limit.*

Proof. Along with (3.199) let us consider the family of equations

$$\frac{dv}{dt} = B(t)v + g \quad ((B, g) \in \omega_{(A,f)}). \tag{3.201}$$

Let us show that every equation of family (3.201) has at most one solution from ω_φ. Suppose the contrary. Then there exist $(B, g) \in \omega_{(A,f)}$ and $\psi_1, \psi_2 \in \omega_\varphi$ that are solutions of (3.201). Note that $\psi = \psi_1 - \psi_2 \not\equiv 0$ is a bounded on $\mathbb{R}$ solution of (3.200). The latter contradicts to the condition of the theorem. So, the conditions of Theorem 3.2.2 are fulfilled and, consequently, φ is compatible in limit. □

Corollary 3.87. *Let φ be a bounded on $\mathbb{R}_+$ solution of* (3.199), *the matrix A and the function f be mutually asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent). If every equation of family* (3.200) *has no nontrivial bounded on $\mathbb{R}$ solutions, then φ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).*

In the case when A and f are asymptotically almost periodic, Corollary 3.87 is a generalization for asymptotic almost periodicity of the known Favard theorem [116] (the first theorem of Favard).

Theorem 3.3.18. *Let A and f be st. L^+. If* (3.198) *is hyperbolic on $\mathbb{R}_+$, then the following statement hold:*

(1) *the homogeneous* (3.199) *has at least one bounded on $\mathbb{R}_+$ solution φ. This solution is given by formula*

$$\varphi(t) := \int_0^{+\infty} G(t,\tau) f(\tau) d\tau, \tag{3.202}$$

where $G(t,\tau)$ is the main Green function [120] for (3.198)*;*

(2) *every bounded on $\mathbb{R}_+$ solution of* (3.199) *is compatible in limit.*

Proof. The first statement of the theorem it follows from [120]. Let us prove the second one. In virtue of Lemma 3.32 and Corollary 3.33 every (3.200) has no nontrivial bounded on $\mathbb{R}$ solutions. According to Theorem 3.3.17 every bounded on $\mathbb{R}_+$ solution of (3.198) is compatible in limit. □

Note that Theorems 3.3.17 and 3.3.18 give sufficient conditions for the existence of bounded on $\mathbb{R}_+$ and compatible in limit solutions of (3.199). However, at first look these theorems essentially differ. The first one states that if there exists a bounded on $\mathbb{R}_+$ solution, then it is compatible in limit, and a priori we do not know whether under the conditions of theorem there exists at least one bounded on $\mathbb{R}_+$ solution. The second theorem states that if its conditions are fulfilled, then there always exists at least one bounded on $\mathbb{R}_+$ solution. The rest of their conclusions coincides. With the reference to the said above there arises the following question. Under the conditions of Theorem 3.3.17, does at least one bounded on $\mathbb{R}_+$ solution of (3.199) exist? Theorem 3.3.8 answers to this question.

Below we investigate the problem of the existence of asymptotically almost periodic solutions of linear differential equations with asymptotically almost periodic coefficients.

Theorem 3.3.19. *Let $A \in C(\mathbb{R}, [E^n])$ be asymptotically almost periodic. The following statements are equivalent:*

(1) *equation* (3.198) *is hyperbolic on $\mathbb{R}_+$;*

(2) *for any asymptotically almost periodic function $f \in C(\mathbb{R}, E^n)$* (3.199) *has at least one asymptotically almost periodic solution.*

Proof. Let (3.198) be hyperbolic on $\mathbb{R}_+$ and $f \in C(\mathbb{R}, E^n)$ be an arbitrary asymptotically almost periodic function. According to Theorem 3.3.18, (3.198) has at least one bounded on $\mathbb{R}_+$ compatible in limit solution φ. From Corollary 3.87 it follows that φ is asymptotically almost periodic. So, from condition (1) follows condition (2). Let us show that the contrary also takes place. Since the matrix A is asymptotically almost periodic, there exist a (unique) almost periodic matrix $P \in \omega_A$ and a matrix $R \in C(\mathbb{R}, [E^n])$ such that

(1) $A(t) = P(t) + R(t)$ for all $t \in \mathbb{R}$;
(2) $\lim_{t\to+\infty} \|R(t)\| = 0$.

Let $g \in C(\mathbb{R}, E^n)$ be an arbitrary almost periodic function. According to the statement, the equation

$$\frac{dy}{dt} = A(t)y + g(t) \tag{3.203}$$

has at least one asymptotically almost periodic solution φ. Since the matrix P is almost periodic and φ is asymptotically almost periodic, there exists a sequence $\{t_n\} \to +\infty$ such that $\{A^{(t_k)}\} \to P$, $g^{(t_k)} \to g$ and $\varphi^{(t_k)} \to q$, where $q \in \omega_\varphi$ is an almost periodic function. Note that q is an almost periodic solution of the equation

$$\frac{dz}{dt} = P(t)z + g(t). \tag{3.204}$$

So, we showed that for any almost periodic function g (3.204) has at least one almost periodic solution. From the results of the work [125] follows that the equation

$$\frac{du}{dt} = P(t)u \tag{3.205}$$

is hyperbolic on $\mathbb{R}$. According to Lemma 3.32 every equation of family (3.200) is hyperbolic on $\mathbb{R}$. By Theorem 3.3.8, (3.199) is hyperbolic on $\mathbb{R}_+$. The theorem is proved. □

Assume

$$M(A) := \lim_{L\to+\infty} \frac{1}{L}\int_0^L A(s)\,ds. \tag{3.206}$$

Theorem 3.3.20. *Let A be asymptotically almost periodic. If the spectrum of the matrix $M(A)$ does not intersect the imaginary axis, then there exists a number $\varepsilon_0 > 0$ such that for every ε, $|\varepsilon| \le \varepsilon_0$ the equation*

$$\frac{dx}{dt} = \varepsilon A(t)x + f(t) \tag{3.207}$$

has at least one asymptotically almost periodic solution for any asymptotically almost periodic function f.

Proof. Let us consider a family of equations

$$\frac{dy}{dt} = \varepsilon B(t)y \quad (B \in \omega_A). \tag{3.208}$$

According to Corollary 1.55, $M(A) = M(P)$, where P is an almost periodic matrix from ω_A such that $\lim_{t\to+\infty} \|A(t) - P(t)\| = 0$. From the results of [125, (see page 258)] it follows the existence of a number $\varepsilon_0 > 0$ such that for every ε, $0 < |\varepsilon| \le \varepsilon_0$, the equation

$$\frac{dz}{dt} = \varepsilon P(t)z \tag{3.209}$$

is hyperbolic on $\mathbb{R}$. Then by Lemma 3.32 every equation of family (3.208) is hyperbolic on $\mathbb{R}$. Consequently, every (3.208) for $0 < |\varepsilon| \le \varepsilon_0$ has no nontrivial bounded on $\mathbb{R}$ solutions. In virtue of Theorem 3.3.8 the equation

$$\frac{dx}{dt} = \varepsilon A(t)x \tag{3.210}$$

is hyperbolic on $\mathbb{R}_+$ and from Theorem 3.3.19 it follows that for $0 < |\varepsilon| \le \varepsilon_0$ (3.207) has at least one asymptotically almost periodic solution for any asymptotically almost periodic function f. □

Let us consider a scalar equation with the asymptotically almost periodic function $a \in C(\mathbb{R}, \mathbb{R})$

$$\frac{dx}{dt} = a(t)x. \tag{3.211}$$

Along with (3.211) consider the nonhomogeneous equation

$$\frac{dy}{dt} = a(t)y + f(t), \tag{3.212}$$

where $f \in C(\mathbb{R}, \mathbb{R})$.

Theorem 3.3.21. *Equation* (3.212) *has at least one asymptotically almost periodic solution for every asymptotically periodic function f if and only if $M(a) \neq 0$ ($M(a)$ is the average value of the function a).*

Proof. Necessity. Let (3.212) have at least one asymptotically almost periodic solution φ for every asymptotically almost periodic function f. According to Theorem 3.3.19, (3.211) is hyperbolic on $\mathbb{R}_+$. For distinctness, let the solutions of (3.211) be bounded on $\mathbb{R}_+$. Then there exist positive numbers N and ν such that

$$|\varphi(t, a, x)| \le N e^{-\nu t}|x| \tag{3.213}$$

for all $t \ge 0$. Since

$$\varphi(t, a, x) = x \exp\left(\int_0^t a(s)ds\right), \tag{3.214}$$

we have

$$\frac{1}{t}\ln|\varphi(t, a, x)| = \frac{\ln|x|}{t} + \frac{1}{t}\int_0^t a(s)ds \tag{3.215}$$

($x \neq 0$). Passing to limit in (3.215) as $t \to +\infty$ and taking into consideration (3.213), we get $M(a) \neq 0$. By analogy, we consider the case when all the solutions of (3.211) are unbounded on $\mathbb{R}_+$.

Sufficiency. Let $M(a) \neq 0$. Then it is easy to verify that $M(a) = M(b)$ for any function $b \in \omega_a$. Let b be an arbitrary function from ω_a. Since $M(b) \neq 0$ and

$$M(b) = \lim_{t \to +\infty} \frac{1}{t} \ln |\varphi(t, b, x)| \tag{3.216}$$

for all $x \neq 0$, it is obvious that the equation

$$\frac{dz}{dt} = b(t)z \tag{3.217}$$

has no nontrivial bounded on $\mathbb{R}$ solutions. Hence, by Theorem 3.3.8, (3.211) is hyperbolic on $\mathbb{R}_+$. To complete the proof of the theorem it is enough to refer to Theorem 3.3.19. □

Theorem 3.3.21 is a generalization of one theorem of Massera (see, i.e., [125, page 43]) for the case of asymptotical almost periodicity.

3.4. Semilinear Differential Equations

In this section we establish the conditions, under which the existence of a compatible in limit solution of a nonlinear equation can be established by the linear terms of the right-hand side of the equation.

Let $\mathfrak{L}$ some set of sequences $\{t_k\} \to +\infty$ and $r > 0$. Denote $C_r(\mathfrak{L}) := \{\varphi : \varphi \in C_b(\mathbb{R}_+, E^n),\ \mathfrak{L} \subseteq \mathfrak{L}_\varphi^{+\infty}$ and $\|\varphi\| \leq r\}$.

Lemma 3.88. *$C_r(\mathfrak{L})$ is a subspace of the metric space $C_b(\mathbb{R}_+, E^n)$.*

Proof. Obviously, to prove the formulated statement it is sufficient to prove that $C_r(\mathfrak{L})$ is closed in $C_b(\mathbb{R}_+, E^n)$. Let $\{\varphi_k\} \subseteq C_r(\mathfrak{L})$ and $\varphi = \lim_{k\to+\infty} \varphi_k$. Let us take an arbitrary $\varepsilon > 0$ and $\{t_k\} \in \mathfrak{L}$. Since $\varphi_k \to \varphi$ in the metric $C_b(\mathbb{R}_+, E^n)$, then $\|\varphi\| \leq r$. Let us show that $\{t_k\} \in \mathfrak{L}_\varphi^{+\infty}$. For $\varepsilon > 0$ there is $k_0 = k_0(\varepsilon)$ such that

$$||\varphi - \varphi_k|| < \frac{\varepsilon}{4} \tag{3.218}$$

for all $k \geq k_0$. Since

$$\begin{aligned} &|\varphi(t + t_l) - \varphi(t + t_r)| \\ &\quad \leq |\varphi(t + t_l) - \varphi_{k_0}(t + t_l)| + |\varphi_{k_0}(t + t_l) - \varphi_{k_0}(t+t_r)| + |\varphi_{k_0}(t+t_r) - \varphi(t+t_r)| \\ &\quad \leq 2||\varphi - \varphi_{k_0}|| + |\varphi_{k_0}(t + t_l) - \varphi_{k_0}(t + t_r)|, \end{aligned} \tag{3.219}$$

then for l and m large enough we have

$$\rho(\varphi^{(t_l)}, \varphi^{(t_m)}) < \varepsilon, \tag{3.220}$$

where ρ is the metric defining an open compact topology in $C(\mathbb{R}_+, E^n)$, that is,

$$\rho(\varphi,\psi) = \sup_{l>0}\Big\{\min\Big\{\max_{0\le t\le l}|\varphi(t)-\psi(t)|, l^{-1}\Big\}\Big\}. \tag{3.221}$$

As the space $C(\mathbb{R}_+, E^n)$ is complete, we conclude that the sequence $\{\varphi^{(t_k)}\}$ is convergent and, consequently, $\mathfrak{L} \subseteq \mathfrak{L}_{\varphi}^{+\infty}$. The lemma is proved. □

Lemma 3.89. *Let $\varphi \in C(\mathbb{R}_+, E^n)$, $F \in C(\mathbb{R}\times E^n, E^n)$ and $\mathfrak{L} \subseteq \mathfrak{L}_{\varphi}^{+\infty} \cap \mathfrak{L}_{F_Q}^{+\infty}$, where $Q := \overline{\varphi(\mathbb{R}_+)}$ and $F_Q := F|_{\mathbb{R}\times Q}$. If F_Q satisfies the condition of Lipschitz with respect to the second variable with the constant $L > 0$, then $\mathfrak{L} \subseteq \mathfrak{L}_{g}^{+\infty}$, where $g(t) := F(t,\varphi(t))$ for all $t \in \mathbb{R}_+$.*

Proof. Let $\{t_n\} \in \mathfrak{L}$. Note that

$$\begin{aligned}
&|g(t+t_l)-g(t+t_r)| \\
&= |F(t+t_l,\varphi(t+t_l))-F(t+t_r,\varphi(t+t_r))| \\
&\le |F(t+t_l,\varphi(t+t_l))-F(t+t_l,\varphi(t+t_r))| + |F(t+t_l,\varphi(t+t_r))-F(t+t_r,\varphi(t+t_r))| \\
&\le L|\varphi(t+t_l)-\varphi(t+t_r)| + \max_{x\in Q}|F(t+t_l,x)-F(t+t_r,x)|.
\end{aligned} \tag{3.222}$$

Passing to limit in inequality (3.222) as $l,r \to +\infty$, we obtain that the sequence $\{g^{(t_k)}\}$ is fundamental in the space $C(\mathbb{R}_+, E^n)$. Since the space $C(\mathbb{R}_+, E^n)$ is complete, the sequence $\{g^{(t_k)}\}$ is convergent, that is, $\{t_k\} \in \mathfrak{L}_{g}^{+\infty}$. The lemma is proved. □

Let us consider a differential equation

$$\frac{dx}{dt} = A(t)x + f(t) + F(t,x), \tag{3.223}$$

where $A \in C(\mathbb{R}, [E^n])$, $f \in C(\mathbb{R}_+, E^n)$ and $F \in C(\mathbb{R}\times W, E^n)$.

Let E_+ be the set of all initial points $x \in E^n$ of solutions from $C_b(\mathbb{R}_+, E^n)$ of (3.198). Then E_+ is a subspace of the space E^n. Denote by P_+ a projector that projects E^n onto E_+.

Lemma 3.90. *Let $A \in C(\mathbb{R}, [E^n])$ be st. L^+. If (3.198) is hyperbolic on $\mathbb{R}_+$, then for any function $f \in C(\mathbb{R}_+, E^n)$ that is st. L^+ (3.199) has the unique compatible in limit solution $\varphi_+ \in C_b(\mathbb{R}_+, E^n)$ satisfying to the condition $P_+\varphi_+(0) = 0$. Besides, there exists a constant $M > 0$ (not depending on f) such that $\|\varphi_+\| \le M\|f\|$.*

Proof. The formulated lemma directly it follows from [126, Lemma 6.3] and Theorem 3.3.18. □

Let φ_+ be a compatible in limit solution of (3.199) the existence of which is guarantied by Lemma 3.90. Assume $Q := \overline{\varphi_+(\mathbb{R}_+)}$ and by Q_r denote a neighborhood of the set $Q \subset E^n$ of radius $r > 0$.

Theorem 3.4.1. *Let $A \in C(\mathbb{R}, [E^n])$, $f \in C(\mathbb{R}_+, E^n)$ and $F \in C(\mathbb{R}_+ \times W, E^n)$. If the following conditions are fulfilled:*

(1) *A, f, and F_{Q_r} are st. L^+;*
(2) *equation* (3.198) *is hyperbolic on $\mathbb{R}_+$;*
(3) *$|F(x,t)| \leq rM^{-1}$ for all $x \in Q_r$ and $t \in \mathbb{R}_+$ (M is the constant, the existence of which is guarantied by Lemma 3.90);*
(4) *F satisfies the condition of Lipschitz with respect to $x \in Q_r$ with the constant of Lipschitz $L < M^{-1}$.*

Then (3.223) *has the unique solution $\varphi \in C(\mathbb{R}_+, Q_r)$ satisfying the condition $P_+\varphi(0) = 0$ and this solution is compatible in limit.*

Proof. In (3.223) let us make a change of the variables: $x(t) = y(t) + \varphi_+(t)$. Then for $y(t)$ we get the differential equation

$$\frac{dy}{dt} = A(t)y + F(t, y + \varphi_+(t)). \tag{3.224}$$

Let $\mathfrak{L} = \mathfrak{L}^{+\infty}(A, f, F_{Q_r})$, where $F_{Q_r} = F|_{\mathbb{R}\times Q_r}$. Define an operator

$$\Phi : C_r(\mathfrak{L}) \longrightarrow C_r(\mathfrak{L}) \tag{3.225}$$

as follows. If $\varphi \in C_r(\mathfrak{L})$, then $\mathfrak{L} \subseteq \mathfrak{L}_{\varphi}^{+\infty}$ and, consequently, $\mathfrak{L} \subseteq \mathfrak{L}_{\varphi+\varphi_+}^{+\infty}$. According to Lemma 3.89 $\mathfrak{L} \subseteq \mathfrak{L}_g^{+\infty}$, where $g(t) := F(t, \varphi(t) + \varphi_+(t))$. By Lemma 3.90, the equation

$$\frac{dz}{dt} = A(t)z + F(t, \varphi(t) + \varphi_+(t)) \tag{3.226}$$

has the unique solution $\psi \in C_b(\mathbb{R}_+, E^n)$ that is compatible in limit (and, consequently, $\mathfrak{L} \subseteq \mathfrak{L}_{\psi}^{+\infty}$) and satisfies the condition $P_+(\psi_+(0)) = 0$. Besides, it is subordinated to the estimate

$$\|\psi\| \leq M\|g\| = M \sup_{t\geq 0} |F(t, \varphi(t) + \varphi_0(t))| \leq M \sup_{t\geq 0} \max_{x\in Q_r} |F(t,x)| \leq MrM^{-1} = r. \tag{3.227}$$

So, $\psi \in C_r(\mathfrak{L})$. Let $\Phi(\varphi) := \psi$. From the said above follows that Φ is well defined. Let us show that the operator Φ is a contraction. In fact, it is easy to note that the function $\psi := \psi_1 - \psi_2 = \Phi(\varphi_1) - \Phi(\varphi_2)$ is a solution of the equation

$$\frac{du}{dt} = A(t)u + F(t, \varphi_1(t) + \varphi_+(t)) - F(t, \varphi_2(t) + \varphi_+(t)), \tag{3.228}$$

with the initial condition $P_+\psi(0) = 0$ and, by Lemma 3.90, it is subordinated to the estimate

$$\begin{aligned} \|\Phi(\varphi_1) - \Phi(\varphi_2)\| &\leq M \sup_{t\geq 0} |F(t, \varphi_1(t) + \varphi_+(t)) - F(t, \varphi_2(t) + \varphi_+(t))| \\ &\leq ML\|\varphi_1 - \varphi_2\| = \alpha\|\varphi_1 - \varphi_2\|. \end{aligned} \tag{3.229}$$

Since $\alpha = ML < MM^{-1} = 1$, then Φ is a contraction and, consequently, there exists the unique function $\overline{\varphi} \in C_r(\mathfrak{L})$ such that $\Phi(\overline{\varphi}) = \overline{\varphi}$. To finish the proof of the theorem it is sufficient to assume that $\varphi := \overline{\varphi} + \varphi_+$ and note that φ is desired solution. The theorem is proved. □

Theorem 3.4.2. *Let A, f be st. L^+ and the following conditions be held:*

(1) *equation* (3.198) *is hyperbolic on* $\mathbb{R}_+$;
(2) $F_{Q_r} = F|_{\mathbb{R}\times Q_r}$ *is st.* L^+;
(3) *F satisfies the condition of Lipschitz with respect to the second variable with the constant* $L > 0$.

Then there exists a number $\varepsilon_0 > 0$ such that for every $|\varepsilon| \leq |\varepsilon_0|$ equation

$$\frac{dx}{dt} = A(t)x + f(t) + \varepsilon F(t,x) \tag{3.230}$$

has the unique compatible in limit solution $\varphi_\varepsilon \in C(\mathbb{R}_+, Q_r)$ satisfying the condition $P_+\varphi_\varepsilon(0) = 0$. Besides, the sequence $\{\varphi_\varepsilon\}$ converges φ_+ as $\varepsilon \to 0$ uniformly with respect to $t \in \mathbb{R}_+$.

Proof. Since the function F_{Q_r} is st. L^+, there exists a constant $N > 0$ such that $|F(t,x)| \leq N$ for all $t \in \mathbb{R}_+$ and $x \in Q_r$. Assume $\varepsilon_0 := \min((LM)^{-1}, r(NM)^{-1})$. Then

$$|\varepsilon F(t,x)| \leq |\varepsilon|\,|F(t,x)| \leq \varepsilon_0 N < r(NM)^{-1}N < rM^{-1} \tag{3.231}$$

for all $t \in \mathbb{R}_+$ and $x \in Q_r$. Obviously, the constant of Lipschits for the function εF is less than M^{-1}. According to Theorem 3.4.1 for every $|\varepsilon| \leq \varepsilon_0$ (3.230) has the unique compatible in limit solution φ_ε satisfying the condition $P_{+\varphi_\varepsilon}(0) = 0$.

Let us estimate the difference $\varphi_\varepsilon(t) - \varphi_+(t) = \psi_\varepsilon(t)$. It is clear that

$$\frac{d\psi_\varepsilon(t)}{dt} = A(t)\psi_\varepsilon(t) + \varepsilon F\big(t, \psi_\varepsilon(t) + \varphi_+(t)\big) \tag{3.232}$$

and, by Lemma 3.90,

$$\|\psi_\varepsilon\| \leq M \sup_{t\geq 0} \big|\varepsilon F\big(t, \psi_\varepsilon(t) + \varphi_+(t)\big)\big| \leq M|\varepsilon| \sup_{t\geq 0}\max_{x\in Q_r} |F(t,x)| \leq M|\varepsilon|N = |\varepsilon|(MN). \tag{3.233}$$

Passing to limit in inequality (3.233) as $\varepsilon \to 0$, we get the necessary statement. The theorem is proved. □

Corollary 3.91. *Let A and f be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic). If the following conditions are fulfilled:*

(1) *F is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic) with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q_r$*;
(2) *equation* (3.198) *is hyperbolic on* $\mathbb{R}_+$;
(3) *F satisfies the condition of Lipschitz with respect to $x \in Q_r$.*

Then there exists $\varepsilon_0 > 0$ such that for every $|\varepsilon| \leq \varepsilon_0$ (3.230) *has the unique asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic) solution $\varphi_\varepsilon \in C(\mathbb{R}_+, Q_r)$ satisfying the condition $P_+\varphi_\varepsilon(0) = 0$. Besides, the sequence $\{\varphi_\varepsilon\}$ converges to φ_+ as $\varepsilon \to 0$ uniformly with respect to $t \in \mathbb{R}_+$*

The formulated statement generalizes the theorem of Biryuk (see, e.g., [116]) for asymptotically almost periodic differential equations.

3.5. Averaging Principle on Semiaxis for Asymptotically Almost Periodic Equations

Let us consider a differential equation

$$\frac{dx}{dt} = \varepsilon f(t,x) \tag{3.234}$$

with the right-hand side $f \in C(\mathbb{R}_+ \times E^n, E^n)$, $\varepsilon > 0$ is a small parameter. Suppose that for every $x \in E^n$ on $\mathbb{R}_+$ there exists an average value with respect to time t of the function f and let

$$f_0(x) := \lim_{L\to+\infty} \frac{1}{L}\int_t^{L+t} f(z,x)\,dz. \tag{3.235}$$

Suppose that the average equation

$$\frac{dx}{dt} = \varepsilon f_0(x) \tag{3.236}$$

has a stationary solution $x_0(t) \equiv x_0$ and let the following conditions be fulfilled:

(C1) the function f is bounded on $\mathbb{R}_+ \times B[x_0,r]$ ($B[x_0,r] := \{x \mid |x-x_0| \le r\}$) and limit (3.235) exists uniformly with respect to $t \in \mathbb{R}_+$ and $x \in B[x_0,r]$;

(C2) there exist $f_x'(t,x)$ and $f_0'(x)$ bounded on $\mathbb{R}_+ \times B[x_0,r]$ and $B[x_0,r]$, respectively;

(C3) the vector-functions $f(t,x)$ and $f_0(x)$ have continuous with respect to $x \in B[x_0,r]$ derivatives and the equality

$$f_0'(x) = \lim_{L\to+\infty} \frac{1}{L}\int_t^{L+t} f_x'(\tau,x)d\tau \tag{3.237}$$

takes place uniformly with respect to x and t;

(C4) the spectrum of the operator $A = f_0'(x_0)$ does not intersect the imaginary axis.

For every $x \in B[x_0,r]$

$$f_0(x+h) - f_0(x) = f_0'(x)h + R(x,h), \tag{3.238}$$

where $|R(x,h)| = o(|h|)$ for every $x, x+h \in B[x_0,r]$. Assume $A = f_0'(x_0)$ and $B(h) := R(x_0,h)$. Then from (3.238) we get

$$f_0(x+h) = Ah + B(h). \tag{3.239}$$

It is possible to show [120] that

$$|\,B(h_1) - B(h_2)\,| \le C(\sigma)\,|\,h_1 - h_2\,| \tag{3.240}$$

for all $|h_1|, |h_2| \le \sigma$ and $C(\sigma) \to 0$ as $\sigma \to 0$.

Transforming (3.234) with the help of (3.239) and of change of variables $h = x - x_0$, we obtain

$$\frac{dh}{dt} = \varepsilon Ah + \varepsilon\big(f(t,x_0+h) - [f_0(x_0+h) - B(h)]\big) \tag{3.241}$$

or, after introducing the denotation $g(t,h) := f(t,x_0+h) - (f_0(x_0+h) - B(h))$,

$$\frac{dh}{dt} = \varepsilon Ah + \varepsilon g(t,h). \tag{3.242}$$

Consider the functions $V(t,h) := f(t,x_0+h) - f_0(x_0+h)$ and $v(t,h;\varepsilon)$, where

$$v(t,h;\varepsilon) := \int_0^{+\infty} V(s+t,h)e^{-\varepsilon s}ds. \tag{3.243}$$

Let us make a change of the variable in (3.242) by the next formula:

$$h := z - \varepsilon v(t,z;\varepsilon). \tag{3.244}$$

Under the made above assumptions replacement (3.244) is invertible ([120]). With the help of replacement (3.244), (3.242) takes the following form:

$$\frac{dz}{d\tau} = Az + F(\tau,z;\varepsilon) \tag{3.245}$$

($\tau = \varepsilon t$). In the same way that in [120] we show that

$$\left|F(\tau,z;\varepsilon) - B(z)\right| = O(\varepsilon) \tag{3.246}$$

for ε small enough and besides, F satisfies the condition of Lipschitz

$$\left|F(\tau,z_1;\varepsilon) - F(\tau,z_2;\varepsilon)\right| \le \mu(\sigma)\left|z_1 - z_2\right| \tag{3.247}$$

for all $|z_1|, |z_2| \le \sigma$ ($\mu(\sigma) \to 0$ as $\sigma \to 0$).

From the results of the work [120] it follows that

$$\lim_{\varepsilon\downarrow 0} \varepsilon v(t,z;\varepsilon) = 0, \qquad \lim_{\varepsilon\downarrow 0} \varepsilon v'_z(t,z;\varepsilon) = 0. \tag{3.248}$$

Applying to (3.245) Theorem 3.4.1 (from the said above it is clear that for (3.245) all the conditions of Theorem 3.4.1 are fulfilled), we obtain that for a sufficiently small $r_0 > 0$ and $\varepsilon_0 > 0$ (3.245) has at least one solution $z(\tau)$ satisfying the condition $|z(\tau)| \le r_0$ for all $\tau \in \mathbb{R}_+$. If with the help of the inverse transformation (3.244), taking into account (3.248), we return to (3.234), then we get the following theorem.

Theorem 3.5.1. *Let the function $f \in C(\mathbb{R} \times E^n, E^n)$ satisfy the conditions (C1)–(C4) and $f_0(x_0) = 0$. If the spectrum of the operator $A = f_0'(x_0)$ does not cross the imaginary axis, then for a sufficiently small $r_0 > 0$ there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ (3.234) has at least one solution $x_\varepsilon(t)$ satisfying the condition*

$$\sup_{t\in\mathbb{R}_+} \left|x_\varepsilon(t) - x_0\right| \le r_0. \tag{3.249}$$

Suppose that besides the enumerated in Theorem 3.5.1 conditions the function $f(t,x)$ and its derivative $f_x'(t,x)$ are asymptotically almost periodic with respect to $t \in \mathbb{R}_+$ uniformly with respect to $x \in B[x_0,r]$.

From the definition of the function $v(t,z;\varepsilon)$ it follows that the function itself and its derivative $v_z'(t,z;\varepsilon)$ are asymptotically almost periodic with respect to $t \in \mathbb{R}_+$ uniformly with respect to $z \in B[0,r_0]$. Since

$$F(\tau,z;\varepsilon) = (I - \varepsilon v_z')^{-1}\left[f_0(x_0+z) + f\left(\frac{\tau}{\varepsilon}, x_0+z-\varepsilon v\right) + \varepsilon v - f\left(\frac{\tau}{\varepsilon}, x_0+z\right)\right] - Az, \tag{3.250}$$

F also is asymptotically almost periodic with respect to $\tau \in \mathbb{R}$ uniformly with respect to $z \in B[0,r_0]$. From the said above and from Corollary 3.91 is what follows.

Theorem 3.5.2. *Let the conditions of Theorem 3.5.1 be held and in addition the function $f(t,x)$ and its derivative $f_x'(t,x)$ be asymptotically almost periodic with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in B[x_0,r]$. Then for $r_0 > 0$ small enough there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$* (3.234) *has at least one asymptotically almost periodic solution $x_\varepsilon(t)$ satisfying condition* (3.249).

3.6. Nonlinear Differential Equations

In this section besides the theorems that follow from general results we will also give some theorems on the existence of asymptotically periodic (resp., asymptotically almost periodic, asymptotically recurrent) solutions that follow from the according theorems about compatible solutions.

Let $\varphi \in C_b(\mathbb{R},E^n)$ and $M \subset C_b(\mathbb{R},E^n)$.

Definition 3.92. Following to [116, page 432], we will say that a function φ is separated in M, if M consists from one function φ or if there exists a number $r > 0$ such that for every function $\psi \in M$ that differs from φ the inequality

$$|\psi(t) - \varphi(t)| \geq r, \tag{3.251}$$

takes place for all $t \in \mathbb{R}$.

Theorem 3.6.1. *Let $\varphi \in C(\mathbb{R}_+,E^n)$ be a bounded on $\mathbb{R}_+$ solution of* (3.1) *and f be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q = \overline{\varphi(\mathbb{R}_+)}$. If all the solutions from ω_φ of every equation of family* (3.6) *are separated in ω_φ, then φ is asymptotically stationary (resp., asymptotically $k_0\tau$-periodic for some natural k_0, asymptotically almost periodic, asymptotically recurrent).*

Proof. Since f is asymptotically recurrent with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q$, then $f_Q = f|_{\mathbb{R}\times Q}$ is st. L^+. By [92, Lemma 3.1.1], the solution φ is st. L^+. Consider the nonautonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ constructed in Example 3.4. Under the conditions of our theorem the point $(\varphi, f_Q) \in X$ is st. L^+. Let us show that all the solutions from $\omega_{(\varphi,f_Q)}$ of every equation of family (3.8) are separated in $\omega_{(\varphi,f_Q)}$. In fact, let $g_Q \in \omega_{f_Q}$ $(\psi_0, g_Q) \in \omega_{(\varphi,f_Q)}$ be a solution of (3.8). Obviously, $\psi_0 \in \omega_\varphi$ is a solution of (3.6) $(g_Q = g|_{\mathbb{R}\times Q})$. According to the condition of the theorem, there exists a number $r = r(g_Q) > 0$ such that for every solution $\psi \in \omega_\varphi$ of (3.6) that differs from ψ_0 inequality (3.251) is held.

Let now $(\psi, g_Q) \in \omega_{(\varphi, f_Q)}$ be an arbitrary, different from (ψ_0, g_Q), solution of (3.6). Then it is clear that the distance between the points (ψ_0, g_Q) and (ψ, g_Q) is not less than r. So, all the solutions from $\omega_{(\varphi, f_Q)}$ of every (3.8) are separated in $\omega_{(\varphi, f_Q)}$. According to Theorems 2.3.2 and 2.4.3 the solution (φ, f_Q) of (3.7) is asymptotically stationary (resp., asymptotically τk_0-periodic for some natural k_0, asymptotically almost periodic, asymptotically recurrent). From the said above follows that φ is asymptotically stationary (resp., asymptotically $k_0\tau$-periodic for some natural k_0, asymptotically almost periodic, asymptotically recurrent). The theorem is proved. □

Remark 3.93. Note that the problem about asymptotical almost periodicity of solutions for differential equations it was studied before, in particular, in the works [34, 73]. In these works for almost periodic right-hand side f and under almost the same conditions that in Theorem 3.6.1 there was proved the asymptotical almost periodicity of the solution φ.

Definition 3.94. According to [30], the solution $\varphi \in C(\mathbb{R}_+, E^n)$ of (3.1) we will call Σ^+-stable, if for every $\varepsilon > 0$ there is δ such that for $t_1, t_2 \in \mathbb{R}_+$ and $Q = \overline{\varphi(\mathbb{R}_+)}$ from the inequalities

$$\rho\big(\varphi^{(t_1)}, \varphi^{(t_2)}\big) < \delta \quad \text{and} \quad \sup_{t\geq 0} \max_{x\in Q} \big| f(t+t_1, x) - f(t+t_2, x)\big| < \delta \tag{3.252}$$

follows the inequality

$$\sup_{t\geq 0} \rho\big(\varphi^{(t+t_1)}, \varphi^{(t+t_2)}\big) < \varepsilon. \tag{3.253}$$

Theorem 3.6.2. *Let φ be a bounded on $\mathbb{R}_+$ solution of* (3.1) *and f be asymptotically almost periodic with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q = \overline{\varphi(\mathbb{R}_+)}$. If φ is Σ^+-stable, then it is asymptotically almost periodic.*

Proof. If φ is a bounded on $\mathbb{R}_+$ solution of (3.1) and f is asymptotically almost periodic with respect to t uniformly with respect to $x \in Q = \overline{\varphi(\mathbb{R}_+)}$, then (φ, f_Q) is a st. L^+ solution of (3.7). It is easy to see that from the Σ^+-stability of the solution φ of (3.1) it follows the Σ^+-stability of the solution (φ, f_Q) of (3.7). By Theorem 2.3.3 the solution (φ, f_Q) is asymptotically almost periodic and, consequently, the solution φ is also asymptotically almost periodic. The theorem is proved. □

Remark 3.95. In the work [106] there is proved a statement analogous to Theorem 3.6.2 with the additional assumption of almost periodicity with respect to t of the right-hand side.

Theorem 3.6.3. *Let φ be a bounded on $\mathbb{R}_+$ solution of* (3.1), *f be asymptotically τ-periodic with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q = \overline{\varphi(\mathbb{R}_+)}$ and $\overline{g}_Q(t,x) := \lim_{k\to+\infty} f_Q(k\tau + t, x)$. If the equation*

$$\frac{dy}{dt} = \overline{g}_Q(t,y) \tag{3.254}$$

admits at most one solution from ω_φ, then the solution φ is asymptotically τ-periodic.

Proof. Under the conditions of the theorem (φ, f_Q) is a st. L^+ solution of (3.7) and equation

$$h(\psi, \overline{g}_Q) = \overline{g}_Q \tag{3.255}$$

has at most one solution from $\omega_{(\varphi,f_Q)}$, where h is a homomorphism of the dynamical systems from Example 3.4. According to Theorem 2.4.1, the solution (φ, f_Q) of (3.7) is asymptotically τ-periodic and, consequently, the solution φ of (3.1) is asymptotically τ-periodic. The theorem is proved. □

Let us consider a differential equation of the second order

$$x'' = f(t,x), \tag{3.256}$$

where $f \in C(\mathbb{R} \times E^n, E^n)$, and give a criterion of the existence of its compatible in limit solutions.

Theorem 3.6.4. *Let $f \in C(\mathbb{R}\times E^n, E^n)$ be continuously differentiable with respect to $x \in E^n$ and let exists $r_0 > 0$ such that*

(1) *$|f(t,x)| \le A(r) < +\infty$ for all $(t,x) \in \mathbb{R}_+ \times B[0,r]$ and $0 \le r \le r_0$;*
(2) *f is asymptotically Poisson stable with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in B[0,r_0]$;*
(3) *there exists positive numbers m and $M(r)$ such that for all $(t,x) \in \mathbb{R}_+ \times B[0,r]$, $0 < r \le r_0$, $mI \le f'_x(t,x) \le M(r)I$ (I is a unit matrix from $[E^n]$) and the matrix $f'_x(t,x)$ is self-adjoint.*

Then for an arbitrary r, $0 \le r \le r_0$, (3.256) *has at least one bounded on $\mathbb{R}_+$ solution φ such that $\mathfrak{L}^{+\infty}_{\hat{f}} \subseteq \mathfrak{L}^{+\infty}_{\varphi}$, where $\hat{f}$ is a restriction of the function f on $\mathbb{R} \times B[0,r]$.*

The proof of Theorem 3.6.4 bases upon the following lemma.

Lemma 3.96. *Let $M > 0$ and $f \in C_b(\mathbb{R}_+, E^n)$. By the formula*

$$\varphi(t) = -\frac{1}{2\sqrt{M}}\left\{e^{\sqrt{M}t}\int_t^{+\infty} e^{-\sqrt{M}\tau} f(\tau)d\tau + e^{-\sqrt{M}\tau}\int_0^t e^{\sqrt{M}\tau} f(\tau)d\tau\right\}, \tag{3.257}$$

there is defined a bounded on $\mathbb{R}_+$ solution of the equation

$$x'' = Mx + f(t), \tag{3.258}$$

and this is a unique solution which may be estimated as follows:

$$\|\varphi\| \leq \frac{1}{M}\|f\|, \tag{3.259}$$

where $\|f\| := \sup\{|f(t)| : t \in \mathbb{R}_+\}$. *If, besides,* f *is asymptotically Poisson stable, then* φ *is compatible in limit.*

Proof. The fact that the function φ defined by equality (3.257) is a solution of (3.258) and can be estimated by (3.259) can be proved by a simple calculation. The second statement follows from Theorem 3.3.18. □

Proof. The proof of Theorem 3.6.4. Let $0 < r \leq r_0$. Assume $\mathfrak{L} = \mathfrak{L}_{\hat{f}}^{+\infty}$, $B_r(\mathfrak{L}) = \{\varphi \mid \varphi \in C_b(\mathbb{R}_+, E^n),\ \|\varphi\| \leq r$, and $\mathfrak{L} \subseteq \mathfrak{L}_{\varphi}^{+\infty}\}$. Further, define an operator Φ from $B_r(\mathfrak{L})$ to $B_r(\mathfrak{L})$ by the equality

$$(\Phi\varphi)(t) = -\frac{1}{2\sqrt{M}}\left\{\int_t^{+\infty} e^{\sqrt{M}(t-\tau)}F(\tau,\varphi(\tau))\,d\tau + \int_0^t e^{-\sqrt{M}(t-\tau)}F(\tau,\varphi(\tau))\,d\tau\right\}, \tag{3.260}$$

where $F(t,x) := f(t,x) - Mx$. Let $\varphi \in B_r(\mathfrak{L})$. Consider a differential equation

$$\frac{d^2x}{dt^2} = Mx + f(t,\varphi(t)) - M\varphi(t). \tag{3.261}$$

Note that $F_x'(t,x) = f_x'(t,x) - MI$, and since $f_x'(t,x)$ is self-adjoint, we have

$$\begin{aligned} \|F_x'(t,x)\| &= \sup_{|\xi|=1} |(F_x'(t,x)\xi,\xi)| = \sup_{|\xi|=1} |(f_x'(t,x)\xi,\xi) - M| \\ &= \sup_{|\xi|=1} |M - (f_x'(t,x)\xi,\xi)| \leq M(r) - m \end{aligned} \tag{3.262}$$

for all $t \in \mathbb{R}_+$ and $x \in B[0,r]$. From inequality (3.262) follows that

$$|F(t,x_1) - F(t,x_2)| \leq (M-m)|x_1 - x_2| \tag{3.263}$$

for all $t \in \mathbb{R}_+$ and $x_1, x_2 \in B[0,r]$.

By Lemma 3.89, $\mathfrak{L} \subset \mathfrak{L}_g^{+\infty}$, where $g(t) := F(t,\varphi(t))$. According to Lemma 3.96, (3.261) has a unique solution $\psi \in C_b(\mathbb{R}_+, E^n)$ such that $\mathfrak{L}_g^{+\infty} \subseteq \mathfrak{L}_{\psi}^{+\infty}$ and, consequently, $\mathfrak{L} \subseteq \mathfrak{L}_{\psi}^{+\infty}$. By the same lemma

$$\begin{aligned} \|\psi\| &\leq \frac{1}{M}\|g\| = \frac{1}{M}\sup_{t\geq 0}|F(t,\varphi(t))| \leq \frac{1}{M}\sup_{t\geq 0}|F(t,\varphi(t)) - F(t,0)| + \frac{1}{M}\sup_{t\geq 0}|F(t,0)| \\ &\leq \frac{1}{M}(M-m)\|\varphi\| + \frac{A(0)}{M} \leq \frac{M-m}{M}r + \frac{A(0)}{M}. \end{aligned} \tag{3.264}$$

From inequality (3.264) it follows that $\psi \in B_r(\mathfrak{L})$, if $mr \geq A(0)$. Put $\psi := \Phi\varphi$. From the above said it follows that $\Phi B_r(\mathfrak{L}) \subseteq B_r(\mathfrak{L})$. In addition, according to Lemma 3.88 $B_r(\mathfrak{L})$ is a closed subspace of the complete metric space $C_b(\mathbb{R}_+, E^n)$. Let us show that

$\Phi : B_r(\mathfrak{L}) \to B_r(\mathfrak{L})$ is a contracting mapping. Let $\varphi_1, \varphi_2 \in B_r(\mathfrak{L})$ and $\psi_i := \Phi\varphi_i$ $(i = 1, 2)$. Then $\psi := \psi_1 - \psi_2$ satisfies equation

$$\frac{d^2x}{dt^2} = Mx + F(t, \varphi_1(t)) - F(t, \varphi_2(t)), \tag{3.265}$$

and can be estimated like this:

$$\|\psi\| = ||\psi_1 - \psi_2|| \le M^{-1} \sup_{t\ge 0} |F(t, \varphi_1(t)) - F(t, \varphi_2(t))| \le \frac{M - m}{M} ||\varphi_1 - \varphi_2||, \tag{3.266}$$

that is,

$$||\Phi\varphi_1 - \Phi\varphi_2|| \le \alpha ||\varphi_1 - \varphi_2|| \tag{3.267}$$

for all $\varphi_1, \varphi_2 \in B_r(\mathfrak{L})$, where $\alpha = M^{-1}(M - m) < 1$. Consequently, there exists a unique fixed point of the operator Φ that, obviously, is the desired solution. The theorem is proved. □

Corollary 3.97. *Let $A \in C_b(\mathbb{R}_+, [E^n])$ be a self-adjoint matrix-function. If there exist positive numbers m and M such that for all $t \in \mathbb{R}_+$*

$$mI \le A(t) \le MI, \tag{3.268}$$

then for any function $f \in C_b(\mathbb{R}_+, E^n)$ the equation

$$x'' = A(t)x + f(t) \tag{3.269}$$

admits at least one bounded on $\mathbb{R}_+$ compatible in limit solution.

Corollary 3.98. *Let the conditions of Theorem 3.6.4 be fulfilled and the function f be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in B[0, r_0]$. Then (3.256) has at least one asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) solution.*

3.7. Bilaterally Asymptotically Almost Periodic Solutions

Let $\varphi \in C(\mathbb{R}, E^n)$ and $(C(\mathbb{R}, E^n), \mathbb{R}, \sigma)$ be a dynamical system of shifts on $C(\mathbb{R}, E^n)$.

Definition 3.99. A function φ is called bilaterally asymptotically stationary (resp., bilaterally asymptotically periodic, bilaterally asymptotically almost periodic, bilaterally asymptotically recurrent), if the motion $\sigma(\cdot, \varphi)$ generated by the function φ in the dynamical system $(C(\mathbb{R}, E^n), \mathbb{R}, \sigma)$ is bilaterally asymptotically stationary (resp., bilaterally asymptotically periodic, bilaterally asymptotically almost periodic, bilaterally asymptotically recurrent), that is, if there exist stationary (resp., periodic, almost periodic, recurrent) functions p_1, $p_2 \in C(\mathbb{R}, E^n)$ such that

$$\varphi(t) = \begin{cases} p_1(t) + r_1(t), & t \in \mathbb{R}_-, \\ p_2(t) + r_2(t), & t \in \mathbb{R}_+, \end{cases} \tag{3.270}$$

where $r_1, r_2 \in C(\mathbb{R}, E^n)$ and $\lim_{t\to-\infty} |r_1(t)| = \lim_{t\to+\infty} |r_2(t)| = 0$. In this case we will use such notation: $(\varphi; p_1, p_2)$.

A typical example of bilaterally asymptotically stationary (constant) function is the function $\varphi(t) = \arctan t$.

Theorem 3.7.1. *Let $A \in C(\mathbb{R}, [E^n])$ and $f \in C(\mathbb{R}, E^n)$ be st. L and $\varphi \in C(\mathbb{R}, E^n)$ be a bounded on $\mathbb{R}$ solution of* (3.199). *If every equation of family*

$$\frac{dy}{dt} = B(t)y, \quad (B \in \Delta_A), \tag{3.271}$$

where $\Delta_A := \omega_A \cup \alpha_A$, has no nonzero bounded on $\mathbb{R}$ solutions, then the solution φ is strongly compatible in limit, that is, $\mathfrak{L}_{(A,f)} \subseteq \mathfrak{L}_\varphi$, where $\mathfrak{L}_\varphi := \{\{t_n\} : \lim_{n\to\pm\infty} |t_n| = +\infty$ and $\{\varphi^{(t_n)}\}$ converges$\}$.

Proof. The formulated statement it follows from Theorem 2.5.2 if we apply it to the nonautonomous dynamical system from Example 3.4 (see the proof of Theorem 3.3.17). □

Corollary 3.100. *Let $A \in C(\mathbb{R}, [E^n])$ and $f \in C(\mathbb{R}, E^n)$ be bilaterally asymptotically stationary (resp., bilaterally asymptotically jointly periodic, bilaterally asymptotically almost periodic, bilaterally asymptotically jointly recurrent) and φ be a bounded on $\mathbb{R}$ solution of* (3.199). *If every equation of family* (3.271) *has no nonzero bounded on $\mathbb{R}$ solutions, then the solution φ is bilaterally asymptotically stationary (bilaterally asymptotically periodic, bilaterally asymptotically almost periodic, bilaterally asymptotically recurrent).*

Definition 3.101. A function $\varphi \in C(\mathbb{R}, E^n)$ is called stationary (resp., periodically, almost periodically, recurrently) homoclinic, if in the dynamical system $(C(\mathbb{R}, E^n), \mathbb{R}, \sigma)$ the motion $\sigma(\cdot, \varphi)$ is stationary (resp., periodically, almost periodically, recurrently) homoclinic, that is, there exists a stationary (resp., periodic, almost periodic, recurrent) function $p \in C(\mathbb{R}, E^n)$ such that

$$\varphi(t) = p(t) + \omega(t) \quad (t \in \mathbb{R}), \tag{3.272}$$

where $\omega \in C(\mathbb{R}, E^n)$ and $\lim_{|t|\to+\infty} |\omega(t)| = 0$. Here we use the notation $(\varphi; p)$.

Definition 3.102. Let $(\varphi_i; p_i)$ $(i = 1, 2)$ be stationary (resp., periodically, almost periodically, recurrently) homoclinic functions from $C(\mathbb{R}, E^{n_i})$ $(i = 1, 2)$. One will say that $(\varphi_1; p_1)$ and $(\varphi_2; p_2)$ are jointly stationary (resp., periodically, almost periodically, recurrently) homoclinic, if the functions $p_1 \in C(\mathbb{R}, E^{n_1})$ and $p_2 \in C(\mathbb{R}, E^{n_2})$ are stationary (resp., jointly periodic, jointly almost periodic, jointly recurrent).

Corollary 3.103. *Let $A \in C(\mathbb{R}, [E^n])$ and $f \in C(\mathbb{R}, E^n)$ be jointly stationary (resp., jointly periodically, jointly almost periodically, jointly recurrently) homoclinic and φ be a bounded on $\mathbb{R}$ solution of* (3.199). *If every equation of family* (3.271) *has no nonzero bounded on $\mathbb{R}$ solutions, then the solution φ is stationary (resp., periodically, almost periodically, recurrently) homoclinic.*

Corollaries 3.100 and 3.103 follow from Theorems 3.7.1 and 2.5.1.

Remark 3.104. Under the conditions of Theorem 3.7.1 and Corollaries 3.100 and 3.103, generally speaking, one cannot assure the existence of at least one bounded on $\mathbb{R}$ solution of (3.199). The next equation confirms it:

$$\frac{dx}{dt} = (\arctan t)x + f(t). \tag{3.273}$$

With reference to the said above the following result of [127] is interesting: let $A \in C(\mathbb{R}, [E^n])$ be st. L and every equation of family (3.271) have no nonzero bounded on $\mathbb{R}$ solutions and $f \in C_b(\mathbb{R}, \mathbb{E}^n)$. For the existence of at least one bounded on $\mathbb{R}$ solution of (3.199) it is necessary and sufficient that

$$\int_{-\infty}^{+\infty} \langle f(t), \psi(t)\rangle dt = 0 \tag{3.274}$$

for every bounded on $\mathbb{R}$ solution ψ of equation

$$\frac{dy}{dt} = -A^*(t)y, \tag{3.275}$$

where $A^*(t)$ is the adjoint matrix for $A(t)$ and $\langle , \rangle$ is the scalar product in E^n.

Denote by $E_0 = \{x \in E^n : \sup\{\varphi(t,x,A) \mid: t \in \mathbb{R}\} < +\infty\}$, where $\varphi(t,x,A) := U(t,A)x$, and P_0 is a projection that projects E^n onto E_0.

Definition 3.105. Recall (see, e.g., [128]) that (3.198) is called weakly regular if for every function $f \in C_b(\mathbb{R}, E^n)$ there exists at least one solution $\varphi \in C_b(\mathbb{R}, E^n)$.

There takes place.

Lemma 3.106. *Let $A \in C(\mathbb{R}, [E^n])$. If A is bounded on $\mathbb{R}$ and* (3.198) *is weakly regular, then for every $f \in C_b(\mathbb{R}, E^n)$ there exists a unique solution $\varphi_0 \in C_b(\mathbb{R}, E^n)$ of* (3.199) *satisfying the following conditions:*

(1) $P_0\varphi_0(0) = 0$;

(2) *there exists a positive constant K (constant of the weakly regularity of* (3.198)*) not depending on f such that $\|\varphi_0\| \le K\|f\|$.*

The formulated statement can be proved in the same way that [126, Lemma 6.3, page 515].

Theorem 3.7.2. *Let a matrix-function $A \in C(\mathbb{R}, [E^n])$ be st. L and* (3.198) *be weakly regular. Then*

(1) *for any bounded on* $\mathbb{R}$ *function* f (3.199) *has at least one bounded on* $\mathbb{R}$ *solution;*
(2) *if* $f \in C(\mathbb{R}, E^n)$ *is st. L, then every bounded on* $\mathbb{R}$ *solution of* (3.199) *is strongly compatible in limit;*
(3) *if* $f \in C(\mathbb{R}, E^n)$ *is st. L, then there exists a unique strongly compatible in limit solution* φ_0 *of* (3.199) *such that* $P_0\varphi_0(0) = 0$ *and* $\|\varphi_0\| \le K\|f\|$, *where K is the constant of weak regularity of* (3.198).

Proof. The first statement of the theorem is obvious. Let us prove the second one. Let $A \in C(\mathbb{R}, [E^n])$, $f \in C(\mathbb{R}, E^n)$ be st. L and (3.198) be weakly regular. Then it is hyperbolic on $\mathbb{R}_+$ and $\mathbb{R}_-$, and, according to Theorem 4.4.1 from [128], every equation of family (3.271) has no nontrivial bounded on $\mathbb{R}$ solutions. According to Theorem 3.7.1 every bounded on $\mathbb{R}$ solution is strongly compatible in limit.

The third statement of the theorem it follows from the second one and from Lemma 3.106. The theorem is proved. □

Theorem 3.7.3 (see [93, 129]). *Let* $A \in C(\mathbb{R}, [E^n])$ *be st. L. Then the following statements hold:*

(1) *if for every* $B \in \omega_A$ (3.200) *has no nontrivial bounded on* $\mathbb{R}$ *solutions, then for every* $B \in \omega_A$ $n^s_A = n^s_B$, *where* $n^s_A = \dim E^s_A$ *and* $E^s_A := \{x \mid x \in E^n,\ |\varphi(t, x, A)| \to 0$ *as* $t \to +\infty\}$;
(2) *if for every* $B \in \alpha_A$ (3.200) *has no nontrivial bounded on* $\mathbb{R}$ *solutions, then for every* $B \in \alpha_A$ $n^u_A = n^u_B$, *where* $n^u_A = \dim E^u_A$ *and* $E^u_A = \{x \mid x \in E^n,\ |\varphi(t, x, A)| \to 0$ *as* $t \to -\infty\}$.

Theorem 3.7.4 (see [125, 128]). *Let* $A \in C(\mathbb{R}, [E^n])$ *be almost periodic and the spectrum of the matrix*

$$\overline{A} := \lim \frac{1}{2T}\int_0^T A(t)dt \tag{3.276}$$

does not intersect the imaginary axis. Then for ε small enough equation

$$\frac{dx}{dt} = \varepsilon A(t)x \tag{3.277}$$

is hyperbolic on $\mathbb{R}$. *Besides,* $n^s_\varepsilon = \overline{n}^s$ *and* $n^u_\varepsilon = \overline{n}^u$ *for sufficiently small ε, where* $n^s_\varepsilon := n^s_{\varepsilon A}$, $n^u_\varepsilon := n^u_{\varepsilon A}$, $\overline{u}^s := n^s_{\overline{A}}$ *and* $\overline{n}^u := n^u_{\overline{A}}$.

Theorem 3.7.5. *Let* $A \in C(\mathbb{R}, [E^n])$ *be bilaterally asymptotically almost periodic and the spectrums of the matrixes*

$$\overline{A}_\pm := \lim_{T\to\pm\infty} \frac{1}{T}\int_0^T A(t)dt \tag{3.278}$$

does not intersect the imaginary axis. Then for sufficiently small ε:

(1) *for arbitrary* $B \in \Delta_A = \omega_A \cup \alpha_A$ *equation*

$$\frac{dy}{dt} = \varepsilon B(t)y \tag{3.279}$$

has no nontrivial bounded on $\mathbb{R}$ *solutions;*

(2) $n^s_\varepsilon = n_+$ $(n_+ := n^s_{\overline{A}_+})$ *and* $n^u_\varepsilon = n_-$ $(n_- := n^u_{\overline{A}_-})$.

Proof. If A is bilaterally asymptotically almost periodic, then there exist almost periodic matrixes $P_+, P_- \in C(\mathbb{R}, [E^n])$ such that

$$A(t) = P_+(t) + R_+(t) \qquad \big(\text{resp., } A(t) = P_-(t) + R_-(t)\big) \tag{3.280}$$

for all $t \in \mathbb{R}_+$ (resp., $t \in \mathbb{R}_-$) and $\lim_{t\to+\infty} \|R_+(t)\| = 0$ (resp., $\lim_{t\to-\infty} \|R_-(t)\| = 0$).

From Theorem 3.7.4 it follows that for ε small enough equation

$$z' = \varepsilon P_\pm(t)z \tag{3.281}$$

is hyperbolic on $\mathbb{R}$ and $n^\alpha_{\varepsilon P_\pm} = n^\alpha_{\overline{A}_\pm} = n^\alpha_\pm$ $(\alpha = s, u)$. Since $\omega_{\varepsilon A} = \omega_{\varepsilon P_+}$ and $\alpha_{\varepsilon A} = \alpha_{\varepsilon P_-}$, the first statement of the theorem it follows from the hyperbolicity on $\mathbb{R}$ of (3.281).

According to Theorem 3.7.3, $n^s_{\varepsilon A} = n^s_{\varepsilon P_+}$ and $n^u_{\varepsilon A} = n^u_{\varepsilon P_-}$. Since for sufficiently small ε we have $n^\alpha_{\varepsilon P_\pm} = n^\alpha_\pm$ $(\alpha = s, u)$, then for the same ε $n^s_{\varepsilon A} = n^s_+$ and $n^u_{\varepsilon A} = n^u_-$. The theorem is proved. □

Corollary 3.107. *Let* $A \in C(\mathbb{R}, [E^n])$ *be bilaterally asymptotically almost periodic and the spectrums of the matrixes* $\overline{A}_+$ *and* $\overline{A}_-$ *does not intersect the imaginary axis. Then*

(1) *if* $n_+ + n_- \geq n$, *then for sufficiently small* ε (3.277) *is weakly regular;*
(2) *if* $n_+ + n_- = n$, *then for sufficiently small* ε (3.277) *is hyperbolic on* $\mathbb{R}$.

Proof. The formulated statement follows from Theorem 3.7.5 and from the results of work [128, (see Problems 32 and 35 on page 107)], and also Theorem 3.3.12. □

Remark 3.108. $n_+ = n^s_{\overline{A}_+}$ (resp., $n_- = n^u_{\overline{A}_-}$) coincides with the number of eigen-values of the matrix $\overline{A}_+$ (resp., $\overline{A}_-$) having negative (resp., positive) real parts.

Theorem 3.7.6. *Let* $A \in C(\mathbb{R}, [E^n])$ *and* $f \in C(\mathbb{R}, E^n)$ *be bilaterally asymptotically almost periodic and the spectrums of the matrixes* $\overline{A}_+$ *and* $\overline{A}_-$ *does not intersect the imaginary axis. Then*

(1) *if* $n_+ + n_- \geq n$, *then for* ε *small enough* (3.207) *has at least one bounded on* $\mathbb{R}$ *solution and every bounded on* $\mathbb{R}$ *solution of* (3.207) *is bilaterally asymptotically almost periodic;*
(2) *if* A *and* f *are jointly stationary (resp., periodically, almost periodically) homoclinic, then for* ε *small enough* (3.207) *has a unique bounded on* $\mathbb{R}$ *solution, that is, stationary (resp., periodically, almost periodically) homoclinic.*

Proof. The theorem follows from Corollary 3.107, Theorem 3.7.2 and Corollaries 3.100 and 3.103. □

Theorem 3.7.7. *Let a matrix-function* $A \in C(\mathbb{R}, [E^n])$ *be bounded on* $\mathbb{R}$, (3.198) *be weakly regular and a function* $F \in C(\mathbb{R} \times E^n, E^n)$ *satisfy the following condition:* $|F(t,x)| \le c(|x|)$ *for all* $t \in \mathbb{R}$ *and* $x \in E^n$*, where* $c : \mathbb{R}_+ \to \mathbb{R}_+$ *is a nondecreasing function. If* $\{r > 0 : Kc(r) \le r\} \neq 0$*, where K is the constant of weak regularity of* (3.198)*, then the equation*

$$\frac{dx}{dt} = A(t)x + F(t,x) \tag{3.282}$$

has at least one bounded on $\mathbb{R}$ *solution.*

Proof. For every function $f \in C_b(\mathbb{R}, E^n)$ (3.200) has a unique solution $\psi \in C_b(\mathbb{R}, E^n)$ such that

$$P_0\psi(0) = 0, \qquad \|\psi\| \le K\|f\|, \tag{3.283}$$

where K is the constant of weak regularity for (3.198). Let $\varphi \in C_b(\mathbb{R}, E^n)$. Consider a differential equation

$$\frac{dy}{dt} = A(t)y + F(t,\varphi(t)). \tag{3.284}$$

Since $|F(t,\varphi(t))| \le c(|\varphi(t)|) \le c(\|\varphi\|)$, the function $f(t) := F(t,\varphi(t))$ is bounded on $\mathbb{R}$ and, consequently, (3.284) has a unique bounded on $\mathbb{R}$ solution ψ_φ satisfying conditions (3.283) and, in particular,

$$||\psi_\varphi|| \le K\|f\| = K\sup_{t\in\mathbb{R}} |F(t,\varphi(t))| \le Kc(\|\varphi\|). \tag{3.285}$$

Define an operator $\Phi : C_b(\mathbb{R}, E^n) \to C_b(\mathbb{R}, E^n)$ as follows: $(\Phi\varphi)(t) := \psi_\varphi(t)$ ($\varphi \in C_b(\mathbb{R}, E^n)$ and $t \in \mathbb{R}$). Let us show that if $r_0 > 0$ satisfies the condition $Kc(r_0) \le r_0$, then the ball $B[0,r_0] = \{\varphi \in C_b(\mathbb{R}, E^n) : \|\varphi\| \le r_0\}$ passes into itself under the mapping Φ. In fact, $\|\Phi_\varphi\| \le Kc(\|\varphi\|) \le Kc(r_0) \le r_0$. Consider now $C_b(\mathbb{R}, E^n)$ as a subset embedded in $C(\mathbb{R}, E^n)$. First of all, note that every ball $B[0,r] \subset C_b(\mathbb{R}, E^n)$ is a convex, bounded, and closed subset of $C(\mathbb{R}, E^n)$.

The mapping $\Phi : B[0,r_0] \to B[0,r_0]$ is continuous in the topology $C(\mathbb{R}, E^n)$. In fact, let $\{\varphi_k\} \subseteq B_{r_0}$ and $\varphi_k \to \varphi$ in $C(\mathbb{R}, E^n)$. Consider the sequence $(\Phi\varphi_k)(t) := \psi_{\varphi_k}(t)$ ($t \in \mathbb{R}$). Note that $f_k(t) := F(t,\varphi_k(t)) \to f(t) = F(t,\varphi(t))$ in the topology $C(\mathbb{R}, E^n)$. If we suppose that it is not so, then there are $\varepsilon_0 > 0$ and $L_0 > 0$ such that

$$\max_{|t|\le L_0} |F(t,\varphi_k(t)) - F(t,\varphi(t))| \ge \varepsilon_0. \tag{3.286}$$

Consequently, there exists $\{t_k\} \subset [-L_0, L_0]$ such that

$$|F(t_k,\varphi_k(t_k)) - F(t_k,\varphi(t_k))| \ge \varepsilon_0. \tag{3.287}$$

Since the sequence $\{t_k\}$ is bounded, it can be considered convergent. Let $t_0 := \lim_{k\to+\infty} t_k$. Since

$$\begin{aligned} |\varphi_k(t_k) - \varphi(t_0)| &\le |\varphi_k(t_k) - \varphi(t_k)| + |\varphi(t_k) - \varphi(t_0)| \\ &\le \max_{|t|\le L_0} |\varphi_k(t) - \varphi(t)| + |\varphi(t_k) - \varphi(t_0)|, \end{aligned} \tag{3.288}$$

then passing into limit as $k \to \infty$ we get $\varphi_k(t_k) \to \varphi(t_0)$. In this case from inequality (3.287) it follows that $\varepsilon_0 \leq 0$. It contradicts to the choice ε_0. So, $f_k \to f$ in $C(\mathbb{R}, E^n)$ and, besides, $\| f_k \| := \sup_{t\in\mathbb{R}} |F(t, \varphi_k(t))| \leq c(\|\varphi_k\|) \leq c(r_0)$, that is, $|f_k(t)| \leq c(r_0)$ $(t \in \mathbb{R})$ and, consequently, $|f(t) \leq c(r_0)$ $(t \in \mathbb{R})$. On the other hand, $\Phi\varphi_k$ is a bounded on $\mathbb{R}$ solution of the equation

$$\frac{du}{dt} = A(t)u + f_k(t). \tag{3.289}$$

In addition, the functions $\Phi\varphi_k$ and their derivatives are uniformly bounded on $\mathbb{R}$ and, hence, the sequence $\{\Phi\varphi_k\}$ is relatively compact in $C(\mathbb{R}, E^n)$. Since $f_k \to f$ in $C(\mathbb{R}, E^n)$, every limiting function of the sequence $\{\Phi\varphi_k\}$ is a bounded on $\mathbb{R}$ solution of (3.199) satisfying conditions (3.283). But in virtue of Lemma 3.106, (3.199) has exactly one bounded on $\mathbb{R}$ solution satisfying conditions (3.283). From this it follows that the sequence $\{\Phi\varphi_k\}$ converges in $C(\mathbb{R}, E^n)$, and the continuity of Φ is established.

Let now prove that the mapping $\Phi : B[0, r_0] \to B[0, r_0]$ is completely continuous in the topology $C(\mathbb{R}, E^n)$. For this aim we note that

$$(\Phi\varphi)'(t) = A(t)(\Phi\varphi)(t) + F(t, \varphi(t)) \tag{3.290}$$

and, consequently,

$$|(\Phi\varphi)'(t)| \leq a|(\Phi\varphi)(t)| + |F(t, \varphi(t))| \leq ar_0 + r_0 \quad (t \in \mathbb{R}), \tag{3.291}$$

where $a := \sup\{\|A(t)\| : t \in \mathbb{R}\}$. From this follows that $\Phi(B[0, r_0])$ is relatively compact in the topology $C(\mathbb{R}, E^n)$. According to the theorem of Tikhonoff-Shauder, the mapping Φ has at least one fixed point $\varphi \in B[0, r_0]$. Obviously, φ is a bounded on $\mathbb{R}$ solution of (3.282). The theorem is proved. □

Theorem 3.7.8. *Let $A \in C(\mathbb{R}, E^n)$ and $F \in C(\mathbb{R} \times E^n, E^n)$ be st. L and the next conditions be satisfied:*

(1) *equation* (3.198) *is weakly regular;*
(2) $|F(t, x)| \leq c(|x|)$ $(t \in \mathbb{R}, x \in E^n)$ *and* $\{r > 0 : Kc(r) \leq r\} \neq \emptyset$, *where* $c : \mathbb{R}_+ \to \mathbb{R}_+$ *is a nondecreasing function, and K is the constant of the weak regularity of* (3.198)*;*
(3) *the restriction F_0 of the function F on $\mathbb{R} \times B[0, r_0]$ satisfies the condition of Lipschitz with respect to the second variable with the constant of Lipschitz $L < K^{-1}$, where r_0 is some positive number satisfying the inequality $Kc(r_0) \leq r_0$.*

Then (3.282) *has at least one bounded on $\mathbb{R}$ weakly compatible in limit solution.*

Proof. Let $\mathfrak{L} := \mathfrak{L}_{(A,F_0)}$ and $C_{r_0}(\mathfrak{L}) := \{\varphi : \varphi \in C_b(\mathbb{R}, E^n), \|\varphi\| \leq r_0\ \mathfrak{L} \subset \mathfrak{L}_\varphi^{+\infty}\}$. Like in Lemma 3.88 it is proved that $C_{r_0}(\mathfrak{L})$ is a closed subset of $C_b(\mathbb{R}, E^n)$. The operator $\Phi : C_b(\mathbb{R}, E^n) \to C_b(\mathbb{R}, E^n)$ defined in the same way that in the proof of Theorem 3.7.3 maps $C_{r_0}(\mathfrak{L})$ into itself. In fact, if $\varphi \in C_{r_0}(\mathfrak{L})$, then, as in Lemma 3.89, it is proved that the function $f(t) = F(t, \varphi(t))$ $(t \in \mathbb{R})$ also belongs to $C_{r_0}(\mathfrak{L})$. According to Theorem 3.7.2,

$\Phi\varphi$ is the unique bounded on $\mathbb{R}$ strongly compatible in limit solution of (3.199) satisfying to conditions (3.283). Since for arbitrary $\varphi_1, \varphi_2 \in C_{r_0}(\mathfrak{L})$

$$(\Phi\varphi_1 - \Phi\varphi_2)'(t) = A(t)[(\Phi\varphi_1 - \Phi\varphi_2)(t)] + F(t,\varphi_1(t)) - F(t,\varphi_2(t)), \tag{3.292}$$

we have

$$||\Phi\varphi_1 - \Phi\varphi_2|| \leq K \sup_{t\in\mathbb{R}} |F(t,\varphi_1(t)) - F(t,\varphi_2(t))| \leq KL||\varphi_1 - \varphi_2||. \tag{3.293}$$

From the last inequality it follows that $\Phi : C_{r_0}(\mathfrak{L}) \to C_{r_0}(\mathfrak{L})$ is a contraction, hence it has a unique stationary point that, in virtue of the proved above facts, is a strongly compatible in limit solution. The theorem is proved. □

Definition 3.109. A function $f \in C(\mathbb{R} \times E^n, E^n)$ is called asymptotically almost periodic (resp., bilaterally, asymptotically, almost periodic) with respect to the variable $t \in \mathbb{R}$ uniformly with respect to x on compacts from E^n, if the motion $\sigma(t, f)$ of the dynamical system $(C(\mathbb{R}\times E^n, E^n), \mathbb{R}, \sigma)$ generated by the function f is asymptotically almost periodic (resp., bilaterally asymptotically almost periodic).

Definition 3.110. Let Q be a compact from E^n. A function $f \in C(\mathbb{R} \times Q, E^n)$ is said to be asymptotically almost periodic with respect to the variable $t \in \mathbb{R}$ uniformly with respect to $x \in Q$ (resp., bilaterally asymptotically almost periodic), if the motion $\sigma(t, f)$ of the dynamical system $(C(\mathbb{R}\times Q, E^n), \mathbb{R}, \sigma)$ generated by the function f is asymptotically almost periodic (resp., bilaterally asymptotically almost periodic).

Corollary 3.111. *If in the conditions of Theorem 3.7.4 the functions $A \in C(\mathbb{R}, [E^n])$ and $F_0 = F|_{\mathbb{R}\times B[0,r_0]}$ are bilaterally asymptotically stationary (resp., asymptotically jointly periodic, asymptotically almost periodic, asymptotically jointly recurrent), then* (3.282) *admits at least one bilaterally asymptotically stationary (resp., asymptotically periodic, asymptotically almost periodic, asymptotically recurrent) solution.*

Corollary 3.112. *If in the conditions of Theorem 3.7.4 the functions $A \in C(\mathbb{R}, [E^n])$ and $F_0 := F|_{\mathbb{R}\times B[0,r_0]}$ are stationary (resp., jointly periodically, almost periodically, jointly recurrently) homoclinic, then* (3.282) *admits at least one stationary (resp., periodically, almost periodically, recurrently) homoclinic solution.*

Theorem 3.7.9. *Let $\varphi \in C(\mathbb{R}, E^n)$ be a bounded on $\mathbb{R}$ solution of* (3.1) *and the function $f \in C(\mathbb{R} \times E^n, E^n)$ be bilaterally asymptotically stationary with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in Q = \overline{\varphi(\mathbb{R})}$ (resp., bilaterally asymptotically periodic, bilaterally asymptotically almost periodic, bilaterally asymptotically recurrent) and the following two conditions be held:*

(1) *for any function $g \in \omega_{f_Q}$ all the solutions of* (3.2) *from ω_φ are separated in ω_φ;*
(2) *for any function $g \in \alpha_{f_Q}$ all the solutions of* (3.2) *from α_φ are separated in α_φ.*

The solution φ is bilaterally asymptotically stationary (resp., bilaterally asymptotically periodic, bilaterally asymptotically almost periodic, bilaterally asymptotically recurrent).

Proof. The formulated theorem it follows from Theorem 3.6.1. □

Theorem 3.7.10. *Let $f \in C(\mathbb{R} \times E^n, E^n)$ and the conditions (C1)–(C4) be fulfilled with respect to $t \in \mathbb{R}$ and $x \in B[x_0, \rho]$. If the spectrum of the matrix $A = f_0'(x_0)$ does not intersect the imaginary axis and f is stationary (resp., τ-periodically, almost periodically) homoclinic with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in B[0, \rho]$ together with its derivative f_x', then for sufficiently small $\rho_0 > 0$ there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ (3.234) has a unique stationary (resp., τ-periodically, almost periodically) homoclinic solution $x_\varepsilon(t)$ satisfying the condition*

$$\sup_{t \in \mathbb{R}} \left| x_\varepsilon(t) - x_0 \right| \leq \rho_0. \tag{3.294}$$

Proof. From equality (3.243) it follows that along with the function $\nu(t, h) := f(t, x_0 + h) - f_0(x_0 + h)$ the function $\nu(t, h, \varepsilon)$ is also stationary (resp., τ-periodically, almost periodically) homoclinic with respect to $t \in \mathbb{R}$ uniformly with respect to $z \in B[0, \rho_0]$. From equality (3.250) it follows that the function F defined by the equality (3.250) is stationary (resp., τ-periodically, almost periodically) homoclinic with respect to $\tau \in \mathbb{R}$ uniformly with respect to $z \in B[0, \rho_0]$ too. From (3.246) and (3.247) follows that we can apply Theorem 3.7.8 and Corollary 3.112 to (3.245). So, (3.245) has at least one stationary (τ-periodically, almost periodically) homoclinic solution $z_\varepsilon(t)$ taking values in the ball $B[0, \rho_0]$. It is easy to notice that condition (3.247) assures the uniqueness of such solution. To complete the proof of the theorem it is sufficient to note that the desired solution of (3.234) is the function

$$x_\varepsilon(t) := x_0 + z_\varepsilon(t) - \varepsilon \nu(t, z_\varepsilon(t), \varepsilon). \tag{3.295}$$

The theorem is proved. □

3.8. Asymptotically Almost Periodic Equations with Convergence

Applying the results of Sections 2.6 and 2.7 to the nonautonomous dynamical system constructed in Example 3.1 generated by (3.1), we obtain series of criterions and tests of convergence of (3.1).

Consider differential (3.1) with the regular right-hand side $f \in C(\mathbb{R} \times E^n, E^n)$.

Definition 3.113. Equation (3.1) is said to be convergent, if the generated by it dynamical system (see Example 3.1 and Corollary 3.2) is convergent.

Let us make this definition precise. Everywhere in this chapter we assume that the right-hand side $f \in C(\mathbb{R} \times E^n, E^n)$ of (3.1) is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) with respect to $t \in \mathbb{R}$ uniformly with respect to x on compact subsets from E^n.

Remark 3.114. According to the given above definition, (3.1) is convergent, if the next conditions are held:

(1) there exists a positive number R such that

$$\limsup_{t\to+\infty} |\varphi(t,u,g)| < R \tag{3.296}$$

for all $u \in E^n$ and $g \in H^+(f)$;

(2) for any $g \in \omega_f$ (3.2) has exactly one bounded on $\mathbb{R}$ solution.

Note that the given by us definition of convergence essentially differs from the conventional (see, i.e., [116]). Usually, the convergence of (3.1) is understood as the presence of a unique bounded on $\mathbb{R}$ globally uniformly asymptotically stable solution of (3.1). In this case the unique bounded solution is called a limit regime of (3.1).

From the results of the works [88, 89, 130] it follows that, if (3.1) possesses the property of convergence in the sense [116, Chapter 4], then every (3.2) possesses this property, too, for every $g \in H^+(f)$ and, consequently, the nonautonomous system generated by (3.1) (see Corollary 3.2) is convergent. At the same time, it is easy to construct examples of nonconvergent in the sense [116, Chapter 4] equations that generate convergent nonautonomous dynamical systems. The thing is that (3.1) can have "limit regime" that is not a solution of (3.1). However, if the right-hand side f of (3.1) does not depend on t (resp., is τ-periodic with respect to t, almost periodic with respect to t, recurrent with respect to t), then the given by us definition coincides with the conventional definition (it follows from the results below).

The following statements take place.

Theorem 3.8.1. *Equation* (3.1) *is convergent if and only if the following conditions are fulfilled:*

(1) *for any $g \in H^+(f)$ every solution $\varphi(t,u,g)$ ($u \in E^n$) of* (3.2) *is bounded on $\mathbb{R}_+$;*

(2) $\lim_{t\to+\infty} |\varphi(t,u_1,g) - \varphi(t,u_2,g)| = 0$ *for all $g \in H^+(f)$ and $u_1, u_2 \in E^n$;*

(3) *for every $\varepsilon > 0$ and $r > 0$ there exists $\delta = \delta(\varepsilon, r) > 0$ such that $|u_1 - u_2| < \delta$ implies $|\varphi(t,u_1,g) - \varphi(t,u_2,g)| < \varepsilon$ for all $t \in \mathbb{R}_+, g \in H^+(f)$ and $u_1, u_2 \in E^n$ such that $|u_1|, |u_2| \le r$.*

Theorem 3.8.2. *For the convergence of* (3.1) *it is necessary and sufficient that the following conditions would hold:*

(1) *for any $g \in H(f)$ every solution $\varphi(t,u,g)$ ($u \in E^n$) of* (3.2) *is bounded on $\mathbb{R}_+$;*

(2) *for arbitrary $g \in H^+(f)$ and $u \in E^n$ the solution $\varphi(t,u,g)$ of* (3.2) *is asymptotically stable, that is, there are held the two following conditions:*

 (a) *for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon,u,g) > 0$ such that $|v - u| < \delta$ implies $|\varphi(t,v,g) - \varphi(t,u,g)| < \varepsilon$ for all $t \in \mathbb{R}_+$;*

 (b) *there exists $\gamma = \gamma(u,g) > 0$ such that $|v-u| < \gamma$ implies* $\lim_{t\to+\infty} |\varphi(t,v,g) - \varphi(t,u,g)| = 0$.

The formulated statements follow from Theorems 2.6.1 and 2.6.2.

Remark 3.115. (1) In the case of almost periodicity of f, Theorem 3.8.1 generalizes and refines the criterion of almost periodic convergence of Zubov [131].

(2) In the case of periodicity f, Theorem 3.8.2 coincides with the theorem of Krasovskii-Pliss [132].

Theorem 3.8.3. *Let* $A \in C(\mathbb{R}, [E^n]), F \in C(\mathbb{R} \times E^n, E^n)$ *and the following conditions be fulfilled:*

(1) $a = \sup\{\|A(t)\| : t \in \mathbb{R}_+\} < +\infty$;
(2) *there are positive numbers* N *and* ν *such that*

$$||U(t,A)U^{-1}(\tau,A)|| \leq Ne^{-\nu(t-\tau)} \quad (t \geq \tau,\ t,\tau \in \mathbb{R}_+); \tag{3.297}$$

(3) $|F(t,u)| \leq M + \varepsilon|u| (u \in E^n,\ t \in \mathbb{R}_+)$ *and* $0 \leq \varepsilon \leq \varepsilon_0 < \nu^2(Na)^{-1}$.

Then the (3.313) *is dissipative, that is, there is a number* $R_0 > 0$ *such that*

$$\limsup_{t\to+\infty} |\varphi(t,v,B,G)| < R_0 \quad (v \in E^n,\ (B,G) \in H^+(A,F)), \tag{3.298}$$

where $\varphi(\cdot,v,B,G)$ *is a solution of the equation*

$$\dot{v} = B(t)v + G(t,v), \tag{3.299}$$

satisfying the initial condition $\varphi(0,v,B,G) = v$.

Proof. For all $B \in H^+(A)$ we will define on E^n a norm $|\cdot|_B$ by the equality

$$|u|_B := \int_0^{+\infty} |U(t,B)u|\, dt. \tag{3.300}$$

As well as in the [109, proof of Theorem 2.39], it is possible to check that

$$\frac{1}{a}|u| \leq |u|_B \leq \frac{N}{\nu}|u| \quad (u \in E^n). \tag{3.301}$$

We put

$$u(t) := |\varphi(t,u,B,G)|_{B_t} = \int_0^{+\infty} |U(s,B_t)\varphi(t,u,B,G)|\, ds. \tag{3.302}$$

Since

$$\varphi(t,u,B,G) = U(t,B)\left(u + \int_0^t U^{-1}(\tau,B)G(\tau,\varphi(\tau,u,B,G))\,d\tau\right), \tag{3.303}$$

then

$$\begin{aligned}
u(t) &= \int_0^{+\infty} \left| U(s+t,B)\left(u + \int_0^t U^{-1}(\tau,B)G(\tau,\varphi(\tau,u,B,G))d\tau\right) \right| ds \\
&\leq \int_0^{+\infty} |U(s+t,B)u|\,ds + \int_0^{+\infty} \left| \int_0^t U(s+t,B)U^{-1}(\tau,B)G(\tau,\varphi(\tau,u,B,G))d\tau \right| ds \\
&\leq \frac{N}{\nu}e^{-\nu t}|u| + \int_0^{+\infty}\int_0^t Ne^{-\nu(s+t-\tau)}(M+\varepsilon|\varphi(\tau,u,B,G)|)d\tau\, ds \\
&= \frac{N}{\nu}e^{-\nu t}|u| + \frac{N}{\nu}e^{-\nu t}\left(\frac{M}{\nu}(e^{\tau t}-1) + \varepsilon\int_0^t |\varphi(\tau,u,B,G)|e^{\nu\tau}d\tau\right) \\
&\leq \frac{N}{\nu}e^{-\nu t}|u| + \frac{NM}{\nu^2}(1-e^{-\nu t}) + \frac{N\varepsilon}{\nu}e^{-\nu t}\int_0^t a|\varphi(\tau,u,B,G)|_{B_\tau}e^{\nu\tau}d\tau \\
&= \frac{N}{\nu}e^{-\nu t}|u| + \frac{NM}{\nu^2}(1-e^{-\nu t}) + \frac{aN\varepsilon}{\nu}e^{-\nu t}\int_0^t u(\tau)e^{\nu\tau}d\tau.
\end{aligned} \tag{3.304}$$

Let $\varphi(t) := u(t)e^{\nu t}$. From the inequality (3.304) follows that

$$\varphi(t) \leq \frac{N}{\nu}|u| + \frac{NM}{\nu^2}(e^{\nu t}-1) + \frac{a\varepsilon N}{\nu}\int_0^t \varphi(\tau)d\tau, \tag{3.305}$$

and by [120, Theorem 9.3] $\varphi(t) \leq \psi(t)$ $(t \in \mathbb{R})$, where ψ is a solution of the integral equation

$$y(t) = \frac{N}{\nu}|u| + \frac{NM}{\nu^2}(e^{\nu t}-1) + \frac{a\varepsilon N}{\nu}\int_0^t y(\tau)d\tau. \tag{3.306}$$

Solving the latter equation, we find that

$$\psi(t) = \left(\frac{N}{\nu}|u| + \frac{NM}{\nu^2 - a\varepsilon N}\right)e^{(a\varepsilon N/\nu)t} + \frac{NM}{\nu^2 - a\varepsilon N}e^{\nu t} \tag{3.307}$$

and, consequently,

$$u(t)e^{\nu t} \leq \left(\frac{N}{\nu}|u| + \frac{NM}{\nu^2 - a\varepsilon N}\right)e^{(a\varepsilon N/\nu)t} + \frac{NM}{\nu^2 - a\varepsilon N}e^{\nu t}. \tag{3.308}$$

From the inequalities (3.304) and (3.308) we obtain

$$|\varphi(t,u,B,G)| \leq a|\varphi(t,u,B,G)|_{B_t} \leq a\left(\frac{N}{\nu}|u| + \frac{NM}{\nu^2 - a\varepsilon N}\right)e^{-((\nu^2 - a\varepsilon N)/\nu)t} + \frac{aNM}{\nu^2 - a\varepsilon N}. \tag{3.309}$$

Therefore,

$$\lim_{t\to+\infty}\sup |\varphi(t,u,B,G)| \leq \frac{aMN}{\nu^2 - a\varepsilon N}, \tag{3.310}$$

$(u \in E^n,\ (B,G) \in H(A,F),$ and $\nu - (a\varepsilon N/\nu) > 0)$. The theorem is proved. □

Theorem 3.8.4. *Suppose that the following conditions are fulfilled:*

(1) $A \in C(\mathbb{R}, [E^n])$, $f \in C(\mathbb{R}, E^n)$, *and* $F \in C(\mathbb{R} \times E^n, E^n)$ *are asymptotically stationary (resp., asymptotically* τ*-periodic, asymptotically almost periodic, jointly asymptotically recurrent);*

(2) *there are positive numbers* N *and* ν *such that*

$$||U(t,A)U^{-1}(\tau,A)|| \leq Ne^{-\nu(t-\tau)} \quad (t \geq \tau,\ t,\tau \in \mathbb{R}_+); \tag{3.311}$$

(3) $|F(t,u)| \leq M + \varepsilon|u|$ $(u \in E^n,\ t \in \mathbb{R}_+)$ *and* $0 \leq \varepsilon \leq \varepsilon_0 < \nu^2(Na)^{-1}$*;*

(4) *the function* F *satisfies the condition of Lipschitz with respect to the space variable uniformly with respect to* $t \in \mathbb{R}$ *with the small enough constant of Lipschitz.*

Then (3.282) *is convergent.*

Proof. According to Theorem 3.8.3 the (3.282) is dissipative. We note that under the condition of Theorem 3.8.4 any ω-limiting equation

$$y' = B(t)y + g(t) + G(t,y) \quad \big((B,g,G) \in \omega_{(A,f,F)}\big) \tag{3.312}$$

admits at most one bounded no $\mathbb{R}$ solution (see [109, Theorem 5.24]). Now to finish the proof it is sufficient to refer to Remark 3.114. □

Theorem 3.8.5. *Let* $f \in C(\mathbb{R} \times E^n, E^n)$ *be asymptotically stationary (resp., asymptotically* τ*-periodic, asymptotically almost periodic, asymptotically recurrent) with respect to* $t \in \mathbb{R}$ *uniformly with respect to* x *on compacts from* E^n*. If there exists* $\alpha > 0$ *such that*

$$\operatorname{Re}\langle (u-v), f(t,u) - f(t,v)\rangle \leq -\alpha|u-v|^2 \tag{3.313}$$

for all $u,v \in E^n$ *and* $t \in \mathbb{R}$*, then* (3.1) *is convergent.*

Proof. Let $Y = H(f)$, $X = E^n \times Y$ and $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ be a nonautonomous dynamical system from Example 3.1. Define on $X\dot{\times}X$ a function V as follows:

$$V\big((u,g),(v,g)\big) = |u-v|. \tag{3.314}$$

The function V, obviously, satisfies the conditions (a)–(c) of Lemma 2.33. Let us show that it satisfies the condition (d) of this lemma too. For this aim we note that for every $g \in H(f)$ we have

$$\operatorname{Re}\langle (u-v), g(t,u) - g(t,v)\rangle \leq -\alpha|u-v|^2 \tag{3.315}$$

for all $u,v \in E^n$ and $t \in \mathbb{R}$. Assume $\varphi(t) := |\varphi(t,u,g) - \varphi(t,v,g)|^2$. Then from (3.315) we get $\varphi' \leq -2\alpha\varphi(t)$ and, consequently,

$$\big|\varphi(t,u,g) - \varphi(t,v,g)\big| \leq e^{-\alpha t}|u-v| \tag{3.316}$$

for all $t \in \mathbb{R}_+, u,v \in E^n$ and $g \in H(f)$. So, the function V satisfies the condition (d) of Lemma 2.33 and, consequently, there exists a unique invariant section $\gamma \in \Gamma(H(f)), E^n \times H(f))$ of the homomorphism h. Besides, from (3.315) it follows that

$$\lim_{t\to+\infty} \big|\varphi(t,u,g) - \varphi(t,v,g)\big| = 0 \tag{3.317}$$

for all $g \in H(f)$ and $u, v \in E^n$. Now to complete the proof of the theorem, like in the previous theorem, it is sufficient to apply Lemma 2.31 to the nonautonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, where $Y = H(f)$. The theorem is proved. □

Remark 3.116. Theorem 3.8.4 remains valid, if condition (3.313) one replace by the following one:

$$\operatorname{Re} \langle W(u - v), f(t, u) - f(t, v) \rangle \leq -\alpha |u - v|^2 \tag{3.318}$$

for all $t \in \mathbb{R}$ and $u, v \in E^n$, where W is some self-adjoint positively defined matrix.

In this case, while proving the theorem, instead of inequality (3.316) we should use the next inequality:

$$|\varphi(t, u, g) - \varphi(t, v, g)| \leq N e^{-\alpha t} |u - v|_w, \tag{3.319}$$

for all $t \in \mathbb{R}_+$ and $u, v \in E^n$, where $|u|_w := \sqrt{\langle Wu, u \rangle}$ and N is some positive constant depending only on the matrix W.

Corollary 3.117. *Let the following conditions be held:*

(1) *$f \in C(\mathbb{R} \times E^n, E^n)$ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) with respect to $t \in \mathbb{R}$ uniformly with respect to x on compact subsets from E^n;*

(2) *f is continuously differentiable with respect to $x \in E^n$;*

(3) *the maximal proper number $\Lambda(t, x)$ of the matrix*

$$f_x'^{*}(t, x)Q + Q f_x'(t, x) \tag{3.320}$$

satisfies the inequality $\Lambda(t, x) \leq -\alpha < 0$, $(t \in \mathbb{R}, x \in E^n)$, where $\alpha > 0$ and Q is some self-adjoint positively defined matrix.

Then (3.1) *is convergent and, in particular, all the solutions of* (3.1) *are asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).*

Proof. The formulated statement it follows from Theorem 3.8.5 and Corollary 3.116. For this it is sufficient to notice that under the conditions of Corollary 3.117, according to [54, Theorem 1], there exist numbers N and ν such that

$$|\varphi(t, u, g) - \varphi(t, v, g)|_Q \leq N e^{-\nu t} |u - v|_Q \tag{3.321}$$

for all $t \in \mathbb{R}$ and $u, v \in E^n$. □

In the case of asymptotical almost periodicity of f, Corollary 3.117 reinforces the general result of the work [23].

Remark 3.118. (a) Theorem 3.8.5 takes place also in the case if in (3.299) the stationary matrix W is replaced by a self-adjoint operator-function $W \in C(\mathbb{R}, [E^n])$ satisfying some additional conditions, analogous to those from [115, Theorem 2].

(b) Theorem 3.8.5 takes place for equations in an arbitrary Hilbert space too.

4 Asymptotically Almost Periodic Distributions and Solutions of Differential Equations

4.1. Bounded on Semiaxis Distributions

For arbitrary $m = 2, 3, \ldots$ by $D^m_{L^1}(\mathbb{R}_+)$ denote the space of functions $\varphi : \mathbb{R}_+ \to E^n$ having $m - 1$ usual derivatives, in this case the derivative $D^{m-1}\varphi$ (the derivative of order $m - 1$) is absolutely continuous, $D^j\varphi \in L^1(\mathbb{R}_+)$ for $0 \leq j \leq m$. By $D^\infty_{L_1}(\mathbb{R}_+)$ denote the space of infinitely differentiable functions, all the derivatives of which belong to $L^1(\mathbb{R}_+)$. In $D^m_{L_1}(\mathbb{R}_+)$, $m < +\infty$, introduce a norm

$$\|\varphi\|_m = \max_{0 \leq j \leq m} \int_0^{+\infty} \left| D^j(t) \right| dt, \tag{4.1}$$

and in $D^\infty_{L_1}(\mathbb{R}_+)$ introduce a locally convex topology defined by the family of norms $\|\cdot\|_m$, $m = 0, 1, 2 \ldots.$

If $\varphi \in D^m_{L^1}(\mathbb{R}_+)$, then

$$\varphi(t) = \int_t^{+\infty} D^j\varphi(s)ds + c, \tag{4.2}$$

where $c \in E^n$. From inequality (4.2) follows that $\lim_{t\to+\infty} \varphi(t) = c$, and since φ is summable, then $c = 0$ and $\lim_{t\to+\infty} \varphi(t) = 0$. If $\varphi \in D^{m+1}_{L^1}(\mathbb{R}_+)$, then the functions $D^j\varphi(t)$, $j = 0, 1, \ldots, m$, tend to zero on infinity and

$$D^j\varphi(t) = \int_t^{+\infty} D^{j+1}\varphi(s)ds \quad (j = \overline{1, m}). \tag{4.3}$$

So,

$$\sup_{t\in\mathbb{R}_+} \left| D^j\varphi(t) \right| \leq \|\varphi\|_{m+1}, \quad (j = \overline{1, m}). \tag{4.4}$$

Lemma 4.1. *$D^m_{L^1}(\mathbb{R}_+)$ $(0 \leq m < +\infty)$ is a Banach space and $D^\infty_{L^1}(\mathbb{R}_+)$ is a space of Fréchet.*

Proof. For $m = 0$ $D^0_{L^1}(\mathbb{R}_+) = L^1(\mathbb{R}_+)$ is a Banach space. Let us show that if for $m \leq q$ the spaces $D^m_{L^1}(\mathbb{R}_+)$ are complete, then $D^{q+1}_{L^1}(\mathbb{R}_+)$ is complete too. In fact, let $\{\varphi_n\}$ be some sequence of Cauchy in $D^{q+1}_{L^1}(\mathbb{R}_+)$. Then $\{\varphi_n\}$ is a Cauchy sequence in $D^q_{L^1}(\mathbb{R}_+)$ and by an inductive supposition there exists such $\varphi \in D^q_{L^1}(\mathbb{R}_+)$ that $\varphi_n \to \varphi$ in $D^q_{L^1}(\mathbb{R}_+)$.

Consider now a sequence of absolutely continuous functions $\{D^q\varphi_n\}$. Since in virtue of (4.4)

$$\sup_{t\in\mathbb{R}_+} |D^q\varphi_n(t) - D^q\varphi_m(t)| \le ||\varphi_n - \varphi_m||_{q+1}, \tag{4.5}$$

then the sequence $\{D^q\varphi_n\}$ converges to some function ψ at every point $t \in \mathbb{R}_+$. By the supposition, the sequence of derivatives $\{D^{q+1}\varphi_n\}$ converges to $L^1(\mathbb{R}_+)$. From here it follows that ψ is absolutely continuous, $\{D^q\varphi_n\}$ converges uniformly to ψ on $\mathbb{R}_+$, and $\{D^{q+1}\varphi_n\}$ converges to $D\psi$ in $L^1(\mathbb{R}_+)$. On the other hand, if $\varphi_n \to \varphi$ in $D^q_{L^1}(\mathbb{R}_+)$, then $D^q\varphi_n \to D^q\varphi$ in $L^1(\mathbb{R}_+)$ and $\psi = D^q\varphi$. Consequently, $\varphi \in D^{q+1}_{L^1}(\mathbb{R}_+)$ and $\varphi_n \to \varphi$ $D^{q+1}_{L^1}(\mathbb{R}_+)$.

Since the topology in the space $D^\infty_{L^1}(\mathbb{R}_+)$ is defined by a countable family of norms, it can be metrizable. Show that it is complete. In fact, let $\{\varphi_n\}$ be some sequence of Cauchy in $D^\infty_{L^1}(\mathbb{R}_+)$. Then $\{\varphi_n\}$ is a sequence of Cauchy in every of the spaces $D^m_{L^1}(\mathbb{R}_+)$ $(m < +\infty)$ and, consequently, there exists $\xi_m \in D^m_{L^1}(\mathbb{R}_+)$ such that $\lim_{n\to+\infty}\varphi_n = \xi_m$ in $D^m_{L^1}(\mathbb{R}_+)$. From here, we have $\xi_0 = \xi_1 = \cdots$ and, therefore, $\xi_0 \in D^m_{L^1}(\mathbb{R}_+)$ and $\varphi_n \to \xi_0$ in $D^\infty_{L^1}(\mathbb{R}_+)$. The lemma is proved. □

Let Q be an open nonempty set in $\mathbb{R}$. Denote by $\mathcal{D}(Q)$ the set of infinitely differentiable functions $\varphi : Q \to E^n$ with a compact support. The convergence in $\mathcal{D}(Q)$ is defined as follows. The sequence $\{\varphi_k\}$ converges to φ, if there exists a compact $K \subset Q$ such that the support $\operatorname{supp}\varphi_k$ of all function φ_k is contained in K and for every m

$$\max_{x\in K} |D^m\varphi_k(x) - D^m\varphi(x)| \longrightarrow 0, \tag{4.6}$$

as $k \to +\infty$. The linear space $\mathcal{D}(Q)$ with the introduced above convergence turns to a locally convex vector topological space [133–135]. By $\mathcal{D}'(Q)$ denote an adjoint space to $\mathcal{D}(Q)$ with a strong topology.

Let $m \in \mathbb{Z}_+$. By $\mathcal{D}^m(Q)$ we will denote the space of the functions $\varphi : Q \to E^n$ with m continuous derivatives and a compact support. If Q is a compact set, then by the equality

$$\|\varphi\|_m = \max_{1\le j\le m}\sup_{x\in Q} |D^j\varphi(x)| \tag{4.7}$$

there is defined a norm on $\mathcal{D}^m(Q)$ and it becomes a Banach space. If Q is a compact, then in $\mathcal{D}(Q)$ we introduce a locally convex topology defined by the family of seminorms $\|\cdot\|_m$.

Let now $Q_1 \subset Q_2$ be compact subsets in Q (Q is not obligatory a compact set). Then $\mathcal{D}^m(Q_1)$ is a subspace of the space $\mathcal{D}^m(Q_2)$ and, hence, in $\mathcal{D}^m(Q)$ we can introduce the topology of a strictly inductive limit of the subspaces $\mathcal{D}^m(Q_1)$.

The space $\mathcal{D}(Q)$ is contained in $\mathcal{D}^m(Q)$. The topology of the space $\mathcal{D}(Q)$ is thinner than the one induced from $\mathcal{D}^m(Q)$. The subspace $\mathcal{D}(Q)$ is dense in $\mathcal{D}^m(Q)$.

Let $Q =]0,+\infty[$ and $\overline{Q} = \mathbb{R}_+ = [0,+\infty[$. Denote by $C^m(Q)$ the family of all the functions $\varphi : Q \to E^n$ having continuous derivatives up to the order m inclusively. The family of all the functions φ from $C^m(Q)$, for which all derivatives $D^m\varphi$ admit a continuous extension onto $\overline{Q}$, denote by $C^m(\overline{Q})$. A norm in $C^m(\overline{Q})$ $(m < +\infty)$ introduce by the formula:

$$\|\varphi\|_{C^m(\overline{Q})} = \max_{0\le j\le m} \sup_{t\in\overline{Q}} |D^j\varphi(t)|. \tag{4.8}$$

Assume $C(Q) := C^0(Q)$ and $C(\overline{Q}) := C^0(\overline{Q})$.

The collection of finite in Q functions of the class $C^m(Q)$ denoted by $C_0^m(Q)$ $(C_0(Q) := C_0^0(Q))$. The family of all the functions of the class $C^m(\overline{Q})$, turning to zero on the boundary Q together with all the derivatives of the order m inclusively, denote by $C_0^m(\overline{Q})$ $(C_0(\overline{Q}) := C_0^0(\overline{Q}))$.

Note that $\mathcal{D}(Q) := C_0^\infty(Q)$ and the space $\mathcal{D}(Q)$ is dense in $\mathcal{D}_{L^1}^m(\mathbb{R}_+)$ $(0 \le m \le +\infty)$.

The space $\beta'^m(\mathbb{R}_+)$, adjoint to $\mathcal{D}_{L^1}^m(\mathbb{R}_+)$ $(0 \le m < +\infty)$, is a Banach space with the norm

$$\|f\|'_m = \sup_{\varphi\in\mathcal{D}_{L^1}^m(\mathbb{R}_+),\,\|\varphi\|\le 1} |\langle f,\varphi\rangle|. \tag{4.9}$$

The restriction of any functional $f \in \beta'^m(\mathbb{R}_+)$ on $\mathcal{D}_{L^1}^{m+1}(\mathbb{R}_+)$ belongs to $\beta'^{m+1}(\mathbb{R}_+)$ and f is completely defined by its restriction, because $C_0^\infty(\mathbb{R}_+)$ is dense in $\mathcal{D}_{L^1}^m(\mathbb{R}_+)$. The operator of restriction establishes an algebraic isomorphism between $\beta'^m(\mathbb{R}_+)$ and some subspace in $\beta'^{m+1}(\mathbb{R}_+)$. So, we can consider that $\beta'^m(\mathbb{R}_+) \subset \beta'^{m+1}(\mathbb{R}_+)$. For $f \in \beta'^m(\mathbb{R}_+)$ it follows that $\|f\|'_{m+1} \le \|f\|'_m$ and, consequently, the topology in $\beta'^m(\mathbb{R}_+)$ is thinner than the one induced from $\beta'^{m+1}(\mathbb{R}_+)$. The operator of restriction establishes an algebraic isomorphism between $\beta'^m(\mathbb{R}_+)$ $(0 \le m < +\infty)$ and some subspace in $\beta'^\infty(\mathbb{R}_+)$, so that we can consider that $\beta'^m(\mathbb{R}_+) \subset \beta'^\infty(\mathbb{R}_+)$. The topology $\beta'^m(\mathbb{R}_+)$ is thinner that one induced from $\beta'^\infty(\mathbb{R}_+)$.

Let $f \in \beta'^\infty(\mathbb{R}_+)$. There exists such neighborhood of zero U in $\mathcal{D}_{L^1}^\infty(\mathbb{R}_+)$ that $|\langle f,\varphi\rangle| \le 1$ for $\varphi \in U$. Then there exists an integer nonnegative number p and such $b > 0$, that $\{\varphi \mid \varphi \in \mathcal{D}_{L^1}^\infty(\mathbb{R}_+), \|\varphi\|_p \le b\} \subset U$. Hence, f is continuous on $\mathcal{D}_{L^1}^\infty(\mathbb{R}_+)$ endowed with the topology induced from $D_{L^1}^p(\mathbb{R}_+)$. Since $\mathcal{D}_{L^1}^\infty(\mathbb{R}_+)$ is dense in $D_{L^1}^p(\mathbb{R}_+)$, then $f \in \beta'^p(\mathbb{R}_+)$. So, we proved the following statement.

Lemma 4.2. $\beta'^\infty(\mathbb{R}_+) = \bigcup_{0\le m<+\infty} \beta'^m(\mathbb{R}_+)$.

From the density of $\mathcal{D}(\mathbb{R}_+)$ in $\mathcal{D}_{L^1}^m(\mathbb{R}_+)$ follows that the restriction on $\mathcal{D}^m(\mathbb{R}_+)$ of some functional $f \in \beta'^m(\mathbb{R}_+)$ defines a distribution in $\mathcal{D}'^m(\mathbb{R}_+)$, and f is well defined by its restriction. Consequently, the space $\beta'^m(\mathbb{R}_+)$ can be identified with some subspace of the space $\mathcal{D}'^m(\mathbb{R}_+)$ of the distributions of the power $\le m$. We will call elements of the space $\beta'^\infty(\mathbb{R}_+)$ bounded on $\mathbb{R}_+$ distributions. Note, that since $\beta'^0(\mathbb{R}_+)$ is adjoint to the space $\mathcal{D}_{L^1}^0(\mathbb{R}_+) = L^1(\mathbb{R}_+)$, then $\beta'^0(\mathbb{R}_+) = L^\infty(\mathbb{R}_+)$. From Lemma 4.2 we conclude that every bounded on $\mathbb{R}_+$ distribution has finite order.

Next we consider multiplicators in the spaces $\beta'^m(\mathbb{R}_+)$. For $\alpha \in \beta^m(\mathbb{R}_+)$, $\varphi \in \mathcal{D}_{L^1}^m(\mathbb{R}_+)$ $(0 \le m < +\infty)$ there takes places the inclusion $\alpha\varphi \in \mathcal{D}_{L^1}^m(\mathbb{R}_+)$ and the bilinear mapping $(\alpha,\varphi) \to \alpha\varphi$ of the product $\beta^m(\mathbb{R}_+) \times D_{L^1}^m(\mathbb{R}_+)$ is continuous in $D_{L^1}^m(\mathbb{R}_+)$. The product αf of the distribution $f \in \beta'^m(\mathbb{R}_+)$ onto the function $\alpha \in \beta^m(\mathbb{R}_+)$ can be defined as follows:

$$\langle \alpha f, \varphi \rangle = \langle f, \alpha\varphi \rangle, \quad (\varphi \in D^m_{L^1}(\mathbb{R}_+)). \tag{4.10}$$

So, functions from $\beta^m(\mathbb{R}_+)$ are multiplicators in $\beta'^m(\mathbb{R}_+)$. For $m < +\infty$ the bilinear mapping $(\alpha, f) \to \alpha f$ from $\beta^m(\mathbb{R}_+) \times \beta'^m(\mathbb{R}_+)$ into $\beta'^m(\mathbb{R}_+)$ is continuous. In fact, let $f_n \to f$, that is, $\langle f_n, \varphi \rangle \to \langle f, \varphi \rangle$ for every $\varphi \in D^m_{L^1}(\mathbb{R}_+)$ and $\alpha_n \to \alpha$ in $\beta^m(\mathbb{R}_+)$. Then

$$\begin{aligned} |\langle \alpha_n f_n, \varphi \rangle - \langle \alpha f, \varphi \rangle| &= |\langle \alpha_n f_n - \alpha_n f, \varphi \rangle + \langle \alpha_n f - \alpha f, \varphi \rangle| \\ &\leq |\langle f_n - f, \alpha_n \varphi \rangle| + |\langle f, (\alpha_n - \alpha)\varphi \rangle| \\ &\leq ||\alpha_n \varphi||_m ||f_n - f||'_m + \|f\|'_m ||(\alpha_n - \alpha)\varphi||_m \end{aligned} \tag{4.11}$$

and, consequently, $\alpha_n f_n \to \alpha f$. For $m = \infty$ this bilinear mapping is separately continuous in any case.

Differentiation in $\beta'^m(\mathbb{R}_+)$ is defined in the usual in the theory of distributions way and has usual properties. If $f \in \beta'^m(\mathbb{R}_+)$, then $Df \in \beta'^{m+1}(\mathbb{R}_+)$, as the operator of differentiation from $\beta'^m(\mathbb{R}_+)$ is adjoint to the operator $-D : \mathcal{D}^{m+1}_{L^1}(\mathbb{R}_+) \to \mathcal{D}^m_{L^1}(\mathbb{R}_+)$. Note that for $\alpha \in \beta'^{m+1}(\mathbb{R}_+)$ and $f \in \beta'^m(\mathbb{R}_+)$ there takes place the equality

$$D(\alpha f) = (D\alpha)f + \alpha(Df). \tag{4.12}$$

Let $h \in \mathbb{R}_+$. The shift $Q + h$ of the open set $Q \subset \mathbb{R}_+$ is open. For $\varphi \in \mathcal{D}^m_{L^1}(Q)$ let $(\tau_h\varphi)(t) := \varphi(t + h)$ so that $\tau_h\varphi \in D^m_{L^1}(Q + h)$. The shift operator of functions on h

$$\tau_h : D^m_{L^1}(Q) \longrightarrow D^m_{L^1}(Q + h) \tag{4.13}$$

is an isomorphism. The shift operator of distributions on h (denote it also by τ_h) we define as an operator from $\beta'^m(Q)$ in $\beta'^m(Q + h)$ by the equality $\langle \tau_h f, \varphi \rangle = \langle f, \tau_h\varphi \rangle$ for all $f \in \beta'^m(Q)$ and $\varphi \in \mathcal{D}^m_{L^1}(Q)$.

4.2. Asymptotically Almost Periodic Distributions

Definition 4.3. A function $\varphi \in \beta^m(\mathbb{R}_+)$ is called asymptotically almost periodic, if $\{\tau_h\varphi \mid h \in \mathbb{R}_+\}$ is a relatively compact set in $\beta^m(\mathbb{R}_+)$.

The space of all asymptotically almost periodic functions from $\in \beta^m(\mathbb{R}_+)$ denote by $\beta^m_{aap}(\mathbb{R}_+)$ and let $\beta^\infty_{aap}(\mathbb{R}_+)$ be the space of all the functions that are asymptotically almost periodic together with all their derivatives.

Definition 4.4. One will say that a distribution $f \in \beta'^\infty(\mathbb{R}_+)$ is asymptotically almost periodic if the shifts $\{\tau_h f \mid h \in \mathbb{R}_+\}$ form a relatively compact set in $\beta'^\infty(\mathbb{R}_+)$.

Let $\beta'^m_{aap}(\mathbb{R}_+)$ be the space of the distributions $f \in \beta'^m(\mathbb{R}_+)$, the shifts of which $\{\tau_h f \mid h \in \mathbb{R}_+\}$ form a relatively compact set in $\beta'^m(\mathbb{R}_+)$. Then $\beta'^m_{aap}(\mathbb{R}_+) \subset \beta'^{m+1}_{aap}(\mathbb{R}_+)$ and $\beta'^m_{aap}(\mathbb{R}_+) \subset \beta'^\infty_{aap}(\mathbb{R}_+)$ $(0 \leq m < +\infty)$.

Lemma 4.5. *The subset $\beta'^m_{aap}(\mathbb{R}_+)$ $(0 \leq m < +\infty)$ is closed in $\beta'^m(\mathbb{R}_+)$.*

Proof. If $f_n \in \beta'^{m}_{aap}(\mathbb{R}_+)$ and $f_n \to f$ in $\beta'^{m}(\mathbb{R}_+)$, then

$$\begin{aligned} \|\tau_h f - \tau_h f_n\|'_m &= \sup_{\varphi\in D^m_{L^1}(\mathbb{R}_+),\, \|\varphi\|_m\le 1} |\langle[\tau_h f - \tau_h f_n],\varphi\rangle| \\ &= \sup_{\varphi\in D^m_{L_1}(\mathbb{R}_+),\, \|\varphi\|_m\le 1} |\langle[f_n - f],\tau_h\varphi\rangle| \\ &\le \sup_{\varphi\in D^m_{L_1}(\mathbb{R}_+),\, \|\varphi\|_m\le 1} |\langle[f - f_n],\varphi\rangle| = \|f - f_n\|'_m. \end{aligned} \tag{4.14}$$

From the last inequality it follows that for every $\varepsilon > 0$ the set of shifts $\{\tau_h f \mid h \in \mathbb{R}_+\}$ possesses a relatively compact ε-net and, hence, $f \in \beta'^{m}_{aap}(\mathbb{R}_+)$. The lemma is proved. □

Denote by V^* the space of measures with a compact support, by V^1 the space of the functions of bounded variation with a compact support, by V^m $(m = 2,3,\dots)$ the space of functions α with some compact support possessing $m-2$ usual derivatives, so that $D^{m-2}\alpha$ absolutely continuous and $D^{m-1}\alpha$ is the function of bounded variation, $V^\infty = \mathcal{D}(\mathbb{R}_+)$. If $\alpha \in V^{m+1}$, then $D\alpha \in V^m$.

Lemma 4.6 (see [8, 136]). *Let $m,q \in \mathbb{Z}_+$ and $\alpha \in V^{m+q}$. Then $\alpha * f \in \beta^q(\mathbb{R}_+)$ for any $f \in \beta'^{m}(\mathbb{R}_+)$ ($*$ is a convolution) and the operator $f \to \alpha * f$ from $\beta'^{m}(\mathbb{R}_+)$ is continuous in $\beta^q(\mathbb{R}_+)$. If $\alpha \in \mathcal{D}(\mathbb{R}_+)$, then $\alpha * f \in \beta^\infty(\mathbb{R}_+)$ for every $f \in \beta'^{\infty}(\mathbb{R}_+)$ and the operator $f \to \alpha * f$ from $\beta'^{\infty}(\mathbb{R}_+)$ is continuous in $\beta^\infty(\mathbb{R}_+)$.*

Let $\alpha \in \mathcal{D}(\mathbb{R}_+)$. From Lemma 4.6 it follows that the convolution operator $f \to \alpha * f$ from $\beta'^{\infty}(\mathbb{R}_+)$ is continuous in $\beta^\infty(\mathbb{R}_+)$. Hence, for $f \in \beta'^{\infty}_{aap}(\mathbb{R}_+)$ the shifts

$$\{\tau_h(\alpha * f) \mid h \ge 0\} = \{\alpha * \{\tau_h f\} \mid h \in \mathbb{R}_+\} \tag{4.15}$$

form a relatively compact set in $\beta^\infty(\mathbb{R}_+)$, that is, $\alpha * f \in \beta^\infty_{aap}(\mathbb{R}_+)$.

Since the operator of differentiation of distributions

$$D : \beta'^{m}(\mathbb{R}_+) \longrightarrow \beta'^{m+1}(\mathbb{R}_+) \tag{4.16}$$

is continuous and permutable together with its shifts, then the derivative of any asymptotically almost periodical distribution is an asymptotically almost periodic distribution.

Corollary 4.7. *If $\alpha \in \beta^m_{aap}(\mathbb{R}_+)$ and $f \in \beta'^{m}_{aap}(\mathbb{R}_+)$, then $\alpha f \in \beta'^{m}_{aap}(\mathbb{R}_+)$.*

Proof. This statements it follows from the fact that the bilinear mapping $(\alpha, f) \to \alpha f$ from $\beta^m(\mathbb{R}_+) \times \beta'^{m}(\mathbb{R}_+)$ is continuous in $\beta'^{m}(\mathbb{R}_+)$ for $m < +\infty$. □

Lemma 4.8 (see [8, 136]). *Let $q \ge 0$ be an integer number and $f \in \mathcal{D}'(\mathbb{R}_+)$. The following statements are equivalent:*

(1) *$f \in \beta'^{m}(\mathbb{R}_+)$ $(0 \le m < +\infty)$;*
(2) *$\alpha * f \in \beta^q(\mathbb{R}_+)$ for every $\alpha \in V^{m+q}$;*
(3) *There exist $\xi \in \beta^0(\mathbb{R}_+)$ and $\eta \in \beta^\infty(\mathbb{R}_+)$ such that $f = D^m\xi + \eta$.*

Lemma 4.9. *Let $q > 0$ be an integer number and $f \in D'(\mathbb{R}_+)$. The next statements are equivalent:*

(1) $f \in {\beta'}^{m}_{aap}(\mathbb{R}_+)$ $(0 \leq m < +\infty)$;

(2) $\alpha * f \in \beta^{q}_{aap}(\mathbb{R}_+)$ *for every* $\alpha \in V^{m+q}$;

(3) *There exist* $\xi \in \beta^{0}_{aap}(\mathbb{R}_+)$ *and* $\eta \in \beta^{\infty}_{aap}(\mathbb{R}_+)$ *such that* $f = D^m\xi + \eta$.

Proof. If $f \in {\beta'}^{m}_{aap}(\mathbb{R}_+)$ and $\alpha \in V^{m+q}$, then by Lemma 4.6 $\alpha * f \in \beta^{q}(\mathbb{R}_+)$. The convolution operator $f \to \alpha * f$ acting from ${\beta'}^{m}(\mathbb{R}_+)$ into $\beta^{q}(\mathbb{R}_+)$ is continuous and there takes place the equality (4.15), therefore $\alpha * f$ is asymptotically almost periodic. So, from (1) it follows (2).

Let (2) take place. In virtue of Lemma 4.9 every distribution $f \in \mathcal{D}'(\mathbb{R}_+)$ can be presented in the form

$$f = D^m\xi + \eta, \tag{4.17}$$

where $\xi \in \beta^{0}(\mathbb{R}_+)$ and $\eta \in \beta^{\infty}(\mathbb{R}_+)$ are given by the formulas

$$\xi = D^q(\alpha_{m+q} * \xi), \qquad \eta = \xi_{m+q} * f. \tag{4.18}$$

From (4.17) and (4.18) it follows that

$$f = D^{m+q}(\alpha_{m+q} * \xi) + \xi_{m+q} * f. \tag{4.19}$$

Since $\alpha_{m+q} \in V^{m+q}$, then $\alpha_{m+q} * f \in \beta^{q}_{aap}(\mathbb{R}_+)$, and, consequently, the function $\xi \in \beta^{0}_{aap}(\mathbb{R}_+)$, since $\psi = \alpha_{m+q} * f \in \beta^{q}_{aap}(\mathbb{R}_+)$. Then $\xi = D^q\psi \in \beta^{0}_{aap}(\mathbb{R}_+)$. From $\xi_{m+q}\mathcal{D}(\mathbb{R}_+)$ it follows that the function $\eta = \xi_{m+q} * f$ is infinitely differentiable. Since $D^j\xi_{m+q} \in V^{m+q}$ for any integer j, the function $D^jh = (D^j\xi_{m+q}) * \xi$ belongs to $\beta^{q}_{aap}(\mathbb{R}_+)$ and hence $\eta \in \beta^{\infty}_{aap}(\mathbb{R}_+)$. So, from (2) it follows (3).

To prove the implication (3)→(1) it is sufficient to prove that $\xi \in \beta^{0}_{aap}(\mathbb{R}_+)$ implies $D^m\xi \in {\beta'}^{m}_{aap}(\mathbb{R}_+)$, since the operator of differentiation D^m of the distributions from ${\beta'}^{0}(\mathbb{R}_+) = \beta^{0}(\mathbb{R}_+)$ is adjoint to the operator of differentiation $(-1)^mD^m : D^{m}_{L^1}(\mathbb{R}_+) \to D^{0}_{L^1}(\mathbb{R}_+)$. Lemma is proved. □

Definition 4.10. One will say that a distribution f is 0-asymptotically almost periodic, if $f \in {\beta'}^{0}_{aap}(\mathbb{R}_+)$ and r-asymptotically almost periodic for $1 \leq r < +\infty$, if $f \in {\beta'}^{r}_{aap}(\mathbb{R}_+)$ and $f \notin {\beta'}^{r-1}_{aap}(\mathbb{R}_+)$.

Lemma 4.11. *For $r \geq 1$ the derivative of r-asymptotically almost periodic distribution $(r+1)$ is asymptotically almost periodic.*

Proof. Let $r \geq 1$ and f be an r-asymptotically almost periodic distribution. Then $Df \in {\beta'}^{r+1}_{aap}(\mathbb{R}_+)$. It remains to show that $Df \notin {\beta'}^{r}_{aap}(\mathbb{R}_+)$. Let $Df \in {\beta'}^{r}_{aap}(\mathbb{R}_+)$, then

$$f = D^{r-1}(\alpha_r * D\xi) + \xi * f. \tag{4.20}$$

If $Df \in {\beta'}^{r}_{aap}(\mathbb{R}_+)$, then we would have $\alpha_r * (D\xi) \in {\beta'}^{r}_{aap}(\mathbb{R}_+)$ and, consequently, $f \in {\beta'}^{r-1}_{aap}(\mathbb{R}_+)$, that contradicts to the choice of f. The lemma is proved. □

4.3. Asymptotically Almost Periodic Solutions of Linear Differential Equations with Distribution Perturbations

Let us consider a differential equation

$$\frac{dx}{dt} = A(t)x, \tag{4.21}$$

and along with (4.21) we will consider the nonhomogeneous equation corresponding to it

$$\frac{dx}{dt} = A(t)x + f(t). \tag{4.22}$$

Let now $A(t) = (\alpha_{ij}(t))_{i,j=1}^{n}$ be such that $\alpha_{ij} \in \beta_{aap}^{m}(\mathbb{R}_+)$ and $f \in \beta_{aap}^{\prime m+1}(\mathbb{R}_+)$.
The following lemma takes place.

Lemma 4.12. *Let* $\{f_k\} \subset L^{\infty}(\mathbb{R}_+)$ *and* $f_k \to f$ *in* L_{loc}^{∞} *(i.e., for any* $l > 0$ $\operatorname{esssup}\{|f_k(t) - f(t)| \mid t \in [0,l]\} \to 0$ *as* $k \to +\infty$*),* $A_k \to A$ *uniformly on compact subsets from* $\mathbb{R}_+$ *and* $x_k \to x$.

Then $\varphi(t, x_k, A_k, f_k) \to \varphi(t, x, A, f)$ *uniformly on compact subsets from* $\mathbb{R}_+$*, where* $\varphi(t, x, A, f)$ *is the solution of* (4.22) *passing through the point* x *as* $t = 0$.

Proof. The formulated statement results from the equality

$$\varphi(t, x_k, A_k, f_k) = U(t, A_k)x_k + \int_0^t U(t, A_k)U^{-1}(\tau, A_k) f_k(\tau)d\tau \tag{4.23}$$

by passing to limit, taking into account the theorem of Lebesgue on the passing to limit under integral and also the properties of the Cauchy operator (see, i.e., [128]). □

Theorem 4.3.1. *Let* (4.21) *be hyperbolic on* $\mathbb{R}_+$, $\alpha_{ij} \in \beta_{aap}^{0}(\mathbb{R}_+)$ *and* $f \in \beta'^{0}_{aap}(\mathbb{R}_+)$. *Then* (4.22) *has at least one asymptotically almost periodic solution* ψ*. This solution can be presented in the form*

$$\psi(t) = \int_0^{+\infty} G_A(t, \tau) f(\tau)d\tau. \tag{4.24}$$

Proof. Formula (4.24) gives a bounded on $\mathbb{R}_+$ solution of (4.22). Let $h_k \to +\infty$, $\{A^{(h_k)}\} \to B$ and $\{f^{(h_k)}\} \to g$. Since ψ is bounded on $\mathbb{R}_+$, the sequence $\{\psi(h_k)\}$ is bounded. Let $h_{k_m} \to +\infty$ be such that $\{\psi(h_{k_m})\}$ converges and $x_0 = \lim_{m\to+\infty} \psi(h_{k_m})$. Then, according to [92, Lemma 3.1.1] $\{\psi(t + h_{k_m})\}$ converges to some function ψ^*, which we can easily see is a bounded $\mathbb{R}$ solution of (3.201). Since (4.21) is hyperbolic on $\mathbb{R}_+$, (3.200) by Lemma 3.32 is hyperbolic on $\mathbb{R}$, and, consequently, (3.201) has a unique bounded on $\mathbb{R}$ solution. Therefore the sequence $\{\psi(t + h_k)\}$ converges to ψ^* uniformly on compact subsets from $\mathbb{R}_+$. Let us show that

$$\sup\{|\psi(t + h_k) - \psi^*(t)| : t \in \mathbb{R}_+\} \longrightarrow 0 \tag{4.25}$$

as $k \to +\infty$. Suppose that it is not so. Then there exist $\epsilon_0 > 0$ and $\{\tau_k\} \subset \mathbb{R}_+$ such that

$$|\psi(\tau_k + h_k) - \psi^*(\tau_k)| \geq \epsilon_0. \tag{4.26}$$

In virtue of asymptotical almost periodicity of f the sequence $\{g^{(\tau_k)}\}$ can be considered converging to $L^\infty_{\text{loc}}(\mathbb{R}_+)$. Let $\overline{g} = \lim_{k\to+\infty} g^{(\tau_k)}$. Then $f^{(\tau_k+h_k)} \to \overline{g}$ in $L^\infty_{\text{loc}}(\mathbb{R}_+)$. Without the loss of generality we can consider that $\{\psi^{*(\tau_k)}\}$ also converges in $L^\infty_{\text{loc}}(\mathbb{R}_+)$ and $\{B^{(\tau_k)}\}$ converges in $C(\mathbb{R}, [E^n])$. Assume $\overline{\psi} = \lim_{k\to+\infty} \psi^{*(\tau_k)}$ and $\overline{B} = \lim_{k\to+\infty} B_{\tau_k}$. It is easy to see that $\overline{\psi}$ is a unique bounded on $\mathbb{R}$ solution of the equation

$$\frac{dy}{dt} = \overline{B}(t)y + \overline{g}(t). \tag{4.27}$$

On the other hand, reasoning in the same way that behind we notice that $\{\psi^{(\tau_k+h_k)}\}$ also converges to $\overline{\psi}$. Hence $\overline{\psi}(0) = \lim_{k\to+\infty} \psi(\tau_k + h_k) = \lim_{k\to+\infty} \psi^*(\tau_k)$. The last contradicts to (4.26). The obtained contradiction shows that there takes place (4.25) and, consequently, ψ is asymptotically almost periodic. The theorem is proved. □

Theorem 4.3.2. *Let* (4.21) *be hyperbolic on* $\mathbb{R}_+$, $A(t) = (\alpha_{ij}(t))^n_{i,j=1}$, $\alpha_{ij} \in \beta^m_{aap}(\mathbb{R}_+)$ *and* $f \in \beta'^{m+1}_{aap}(\mathbb{R}_+)$. *Then* (4.22) *has at least one asymptotically almost periodic generalized solution* $\psi \in \beta'^m_{aap}(\mathbb{R}_+)$.

Proof. Since the derivative of any distribution from $\beta'^m_{aap}(\mathbb{R}_+)$ belongs to $\beta'^{m+1}_{aap}(\mathbb{R}_+)$, then

$$L(\beta'^m_{aap}(\mathbb{R}_+)) \subset \beta'^{m+1}_{aap}(\mathbb{R}_+), \tag{4.28}$$

where $(Lf,\varphi) = (f, L^*\varphi)$ and $(L^*\varphi)(t) = \varphi'(t) + A^*(t)\varphi(t)$ for all $f \in \beta'^m(\mathbb{R}_+)$ and $\varphi \in \beta^m(\mathbb{R}_+)$.

Let us prove the inverse inclusion. For that it is sufficient to show that (4.22) has a solution in $\beta'^m_{aap}(\mathbb{R}_+)$, for any $f \in \beta'^{m+1}_{aap}(\mathbb{R}_+)$. Let $f \in \beta'^{m+1}_{aap}(\mathbb{R}_+)$ be r-asymptotically almost periodic so that $r \leq m+1$. If $r = 0$, then by Theorem 4.3.1, (4.22) has at least one asymptotically almost periodic solution.

Show that if for an arbitrary r-asymptotically almost periodic $f \in \beta'^{m+1}_{aap}(\mathbb{R}_+)$ for all $r \leq q-1$ (4.22) has a solution in $\beta'^m_{aap}(\mathbb{R}_+)$, then it has solutions in $\beta'^{m+1}_{aap}(\mathbb{R}_+)$ also in the case when f is q-asymptotically almost periodic. In fact, according to Lemma 4.9 there exist $\xi \in \beta^0_{aap}(\mathbb{R}_+)$ and $\eta \in \beta^\infty_{aap}(\mathbb{R}_+)$ such that $f = D^q\xi + \eta$. The equation $Lx = \eta$ has and asymptotically almost periodic solution. Making a replacement of variables $x = z + D^{q-1}\xi$ in the equation $Lx = D^q\xi$, we get an equivalent equation

$$Lz = -A(t)D^{q-1}z, \tag{4.29}$$

where $A(t)D^{q-1}z$ has the rank $\leq q-1$. Hence the equation $Lx = f$ has solutions in $\beta'^m_{aap}(\mathbb{R}_+)$ for any $f \in \beta'^{m+1}_{aap}(\mathbb{R}_+)$. So, $L(\beta'^m_{aap}(\mathbb{R}_+)) = \beta'^{m+1}_{aap}(\mathbb{R}_+)$. The theorem is proved. □

4.4. Asymptotically Almost Periodic Distributions

Recall that $AP(\mathbb{R}_+)$ defines the set of all asymptotically almost periodic functions from $C(\mathbb{R}_+, E^n)$, that is, functions $\varphi \in C(\mathbb{R}_+, E^n)$ that can be presented in the form of the sum $p + \omega$, where $p \in C(\mathbb{R}_+, E^n)$ is almost periodic and $\lim_{t\to+\infty} |\omega(t)| = 0$.

By $AP^m(\mathbb{R}_+)$ let us denote the set of all m-times continuously differentiable functions from $C(\mathbb{R}_+, E^n)$ that are asymptotically almost periodic together with their derivatives up to the order m inclusively, that is,

$$AP^m(\mathbb{R}_+) = \{\varphi \mid D^k\varphi \in AP(\mathbb{R}_+),\ k = \overline{0, m}\}. \tag{4.30}$$

By the equality

$$\|\varphi\|_m = \max_{0\le j\le m} \sup_{t\in\mathbb{R}_+} |D^j\varphi(t)|, \tag{4.31}$$

there is defined a norm on $AP^m(\mathbb{R}_+)$, and with this norm $AP^m(\mathbb{R}_+)$ is a Banach space.

The convolution of two functions $\varphi, \psi \in AP^m(\mathbb{R}_+)$ define by the equality

$$(\varphi * \psi)(t) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \langle \varphi(t+s), \psi(s)\rangle\, ds, \tag{4.32}$$

that is, $(\varphi * \psi)(t) = M\{\langle \varphi(t+s), \psi(s)\rangle\}$. There takes place.

Lemma 4.13. *For $\varphi, \psi \in AP^m(\mathbb{R}_+)$*

$$\varphi * \psi = p * q, \tag{4.33}$$

where p and q are the main parts of the functions φ and ψ, respectively.

Proof. Note that

$$\begin{aligned}\langle \varphi(t+s), \psi(s)\rangle &= \langle p(t+s) + \omega(t+s), q(s) + \overline{\omega}(s)\rangle \\ &= \langle p(t+s), q(s)\rangle + \langle \omega(t+s), q(s)\rangle + \langle p(t+s), \overline{\omega}(s)\rangle + \langle \omega(t+s), \overline{\omega}(s)\rangle \\ &= \langle p(t+s), q(s)\rangle + \overline{\overline{\omega}}(t,s),\end{aligned} \tag{4.34}$$

where $\overline{\overline{\omega}}(t,s) = \langle \omega(t+s), q(s)\rangle + \langle p(t+s), \overline{\omega}(s)\rangle + \langle \omega(t+s), \overline{\omega}(s)\rangle$ and, consequently, $\overline{\overline{\omega}}(t,s) \to 0$ as $s \to +\infty$ (for every $t \in \mathbb{R}_+$). From (4.32) and (4.34) it follows equality (4.33). The lemma is proved. □

Lemma 4.14. *Let $\varphi, \psi \in AP^m(\mathbb{R}_+)$ $(m \ge 1)$. Then*

$$D^j(\varphi * \psi) = (D^j\varphi) * \psi. \tag{4.35}$$

Proof. Since $h^{-1}[\varphi(t+h) - \varphi(t)]$ $(t \in \mathbb{R}_+)$ as $h \downarrow 0$ has the limit $\varphi'(t)$ (uniformly with respect to $t \in \mathbb{R}_+$), then

$$\left\langle \frac{\varphi(s+t+h) - \varphi(s+t)}{h}, \psi(t)\right\rangle \longrightarrow \langle \varphi'(s+t), \psi(t)\rangle \tag{4.36}$$

as $h \downarrow 0$ (uniformly with respect to $s \in \mathbb{R}_+$ for every $t \in \mathbb{R}_+$) and hence

$$\varphi' * \psi = \lim_{h \to 0} \varphi_h * \psi, \tag{4.37}$$

where $\varphi_h(t) := h^{-1}[\varphi(t+h) - \varphi(t)]$.

On the other hand, from the relation

$$\left\langle \frac{(\varphi(s+t+h) - \varphi(s+t))}{h}, \psi(s) \right\rangle = \frac{[\langle \varphi(s+t+h), \psi(s)\rangle - \langle \varphi(s+t), \psi(s)\rangle]}{h}, \tag{4.38}$$

it follows that

$$(\varphi_h * \psi)(t) = \frac{[M_s\{\langle \varphi(s+t+h), \psi(s)\rangle\} - M_s\{\langle \varphi(s+t), \psi(s)\rangle\}]}{h}. \tag{4.39}$$

Passing to the limit in (4.39) and taking into consideration (4.37), we obtain

$$(\varphi * \psi)' = \varphi' * \psi. \tag{4.40}$$

Repeating this process we get the desired relation. Lemma is proved. □

Denote by $AP'^m(\mathbb{R}_+)$ the space adjoint to $AP^m(\mathbb{R}_+)$. The restriction of any functional $f \in AP'^m(\mathbb{R}_+)$ $(m < +\infty)$ on $AP^{m+1}(\mathbb{R}_+)$ belongs to $AP'^{m+1}(\mathbb{R}_+)$ and f is well defined by its restriction. The restriction operator establishes an isomorphism between $AP'^m(\mathbb{R}_+)$ and some subspace in $AP'^{m+1}(\mathbb{R}_+)$. That is why we can consider that $AP'^m(\mathbb{R}_+) \subset AP'^{m+1}(\mathbb{R}_+)$, and as $m < +\infty$ $AP'^m(\mathbb{R}_+) \subset AP'^\infty(\mathbb{R}_+)$.

Lemma 4.15. $AP'^\infty(\mathbb{R}_+) = \bigcup_{0 \le m < +\infty} AP'^m(\mathbb{R}_+)$.

Proof. Behind was noted that $AP'^m(\mathbb{R}_+) \subset AP'^\infty(\mathbb{R}_+)$. Further, if $f \in AP'^\infty(\mathbb{R}_+)$, then in $AP^\infty(\mathbb{R}_+)$ there exists a neighborhood of zero U such that $|\langle f, \varphi\rangle| \le 1$ for $\varphi \in U$. Hence, there exists an integer nonnegative number m and $b > 0$ such that from $\varphi \in AP^\infty(\mathbb{R}_+)$, $\|\varphi\|_m \le b$ it follows that $\varphi \in U$. Therefore the functional f is continuous on the space $AP^\infty(\mathbb{R}_+)$ endowed with the topology induced from $AP^m(\mathbb{R}_+)$. Since $AP^\infty(\mathbb{R}_+)$ is dense in $AP^m(\mathbb{R}_+)$, then $f \in AP'^m(\mathbb{R}_+)$. The lemma is proved. □

In $AP^m(\mathbb{R}_+)$ derivative, shifts and product by the function are defined in a usual way: the operator D of differentiation of functionals from $AP'^m(\mathbb{R}_+)$ is the operator adjoint to the operator $-D : AP^{m+1}(\mathbb{R}_+) \to AP^m(\mathbb{R}_+)$; the operator τ_h of shift of functionals from $AP'^m(\mathbb{R}_+)$ is the operator adjoint to the shift operator $\tau_h : AP^m(\mathbb{R}_+) \to AP^m(\mathbb{R}_+)$; the multiplicators on $AP'^m(\mathbb{R}_+)$ are the functions $\alpha \in AP^m(\mathbb{R}_+)$, and the product operator of functionals from $AP'^m(\mathbb{R}_+)$ on α is the operator adjoint to the product operator on α in the space $AP^m(\mathbb{R}_+)$.

The action of the asymptotically almost periodic function distribution $g \in AP'^\infty(\mathbb{R}_+)$ on the function $\varphi \in AP^\infty(\mathbb{R}_+)$ we define by the equality

$$\langle g * \varphi\rangle(t) = \langle g, \varphi^{(t)}\rangle, \tag{4.41}$$

that is, $\langle g * \varphi\rangle : t \to \langle g, \varphi^{(t)}\rangle = \psi(t)$.

Note that $AP'^\infty(\mathbb{R}_+)$ is not a normed space (The space $AP^\infty(\mathbb{R}_+)$ is countable normed: $\|\cdot\|_k$, $k = 0, 1, \dots$).

Lemma 4.16. *The function* ψ *defined by rule* (4.41) *belongs to* $AP^{\infty}(\mathbb{R}_+)$, *if* $g \in AP'^{\infty}(\mathbb{R}_+)$.

Proof. Note that

$$\left|\psi(t+\tau) - \psi(t)\right| = \left|\langle g, \varphi^{(t+\tau)}\rangle - \langle g, \varphi^{(t)}\rangle\right| = \left|\langle g, \varphi^{(t+\tau)} - \varphi^{(t)}\rangle\right|. \tag{4.42}$$

From Lemma 4.15 for $g \in AP'^{\infty}(\mathbb{R}_+)$ it follows the existence of $m \geq 0$ such that $g \in AP'^{m}(\mathbb{R}_+)$, hence we have

$$\left|\langle g, \varphi^{(t+\tau)} - \varphi^{(t)}\rangle\right| \leq \|g\|'_m \|\varphi^{(t+\tau)} - \varphi^{(t)}\| < \varepsilon, \tag{4.43}$$

only if $\|\varphi^{(t+\tau)} - \varphi^{(t)}\| < \varepsilon/\|g\|'_m$. From equality (4.42) for the first derivative ψ we get the equality $|\psi'(t+\tau) - \psi'(t)| = |\langle g, \varphi'^{(t+\tau)} - \varphi'^{(t)}\rangle|$ and, consequently,

$$\left|\psi'(t+\tau) - \psi'(t)\right| \leq \|g\|'_m \|\varphi'^{(t+\tau)} - \varphi'^{(t)}\| < \varepsilon, \tag{4.44}$$

if $\|\varphi'^{(t+\tau)} - \varphi'^{(t)}\| < \varepsilon/\|g\|'_m$. Repeating this process further we obtain the necessary statement. The lemma is proved. □

Let $f, g \in AP'^{\infty}(\mathbb{R}_+)$, then the convolution $f * g$ is defined by the equality $\langle g * f, \varphi\rangle = \langle f, g * \varphi\rangle$ $(\varphi \in AP^{\infty}(\mathbb{R}_+))$.

Lemma 4.17. *Let* $f, g \in AP'^{\infty}(\mathbb{R}_+)$. *Then*

(1) $\tau_h(g * f) = (\tau_h g) * f$;
(2) $D(g * f) = (Dg) * f$.

Proof. For $\varphi \in AP^{\infty}(\mathbb{R}_+)$ we have

$$\langle \tau_h(g * f), \varphi\rangle = \langle g * f, \tau_h\varphi\rangle = \langle f, \langle g, \tau_h\varphi\rangle\rangle = \langle f, \langle \tau_h g, \varphi\rangle\rangle = \langle (\tau_h g) * f, \varphi\rangle. \tag{4.45}$$

Let now prove the second statement. Let $\varphi \in AP^{\infty}(\mathbb{R}_+)$, then

$$\langle D(g * f), \varphi\rangle = \langle g * f, -D\varphi\rangle = \langle f, \langle g, -D\varphi\rangle\rangle = \langle f, \langle Dg, \varphi\rangle\rangle = \langle (Dg) * f, \varphi\rangle. \tag{4.46}$$

The lemma is proved. □

4.5. Solvability of the Equation $x' = A(t)x + f(t)$ in the Class of Asymptotically Almost Periodic Distributions $AP'^{m}(\mathbb{R}_+)$

In this section we will consider (4.22) with $A(t) = (\alpha_{ij}(t))_{i,j=1}^{n}$, where $\alpha_{ij} \in AP^{m}(\mathbb{R}_+)$, and $f \in AP'^{m+1}(\mathbb{R}_+)$.

Define the operator $L : AP^{m+1}(\mathbb{R}_+) \to AP^{m}(\mathbb{R}_+)$ by the equality

$$(Lx)(t) = \frac{dx}{dt}(t) - A(t)x(t). \tag{4.47}$$

By L^* denote the operator formally adjoint to L, defined by the equation

$$(L^* y)(t) = \frac{dy}{dt}(t) + A^*(t)y(t). \tag{4.48}$$

Lemma 4.18. *There takes place the equality*

$$M\{\langle x(t), -\dot{\varphi}(t) - A^*(t)\varphi(t)\rangle\} = M\{\langle f(t), \varphi(t)\rangle\}, \tag{4.49}$$

or, which is the same,

$$\langle x, L^*\varphi\rangle = \langle f, \varphi\rangle \tag{4.50}$$

for all $\varphi \in AP^{m+1}(\mathbb{R}_+)$ *and* $x \in AP^m(\mathbb{R}_+)$ $(m \geq 1)$.

Proof. Let $\varphi \in AP^{m+1}(\mathbb{R}_+)$ and $x \in AP^m(\mathbb{R}_+)$. Then

$$\frac{1}{T}\int_0^T \langle \dot{x}(t) - A(t)x(t), \varphi(t)\rangle dt = \frac{1}{T}\int_0^T \langle f(t), \varphi(t)\rangle dt. \tag{4.51}$$

Since

$$\int_0^T \langle \dot{x}(t), \varphi(t)\rangle dt = \langle x(t), \varphi(t)\rangle|_0^T + \int_0^T \langle x(t), -\dot{\varphi}(t)\rangle dt, \tag{4.52}$$

then

$$\frac{1}{T}\int_0^T \langle x(t), -\dot{\varphi}(t) - A^*(t)\varphi(t)\rangle dt + \frac{1}{T}\langle x(t), \varphi(t)\rangle|_0^T = \frac{1}{T}\int_0^T \langle f(t), \varphi(t)\rangle dt. \tag{4.53}$$

Note that $|\langle x(T), \varphi(T)\rangle - \langle x(0), \varphi(0)\rangle| \leq M$ and hence

$$\left|\frac{1}{T}\langle x(t), \varphi(t)\rangle\right|_0^T \leq \frac{M}{T} \longrightarrow 0 \tag{4.54}$$

as $T \to +\infty$. From that it follows equality (4.49). The lemma is proved. □

Let now $f \in AP'^{m+1}(\mathbb{R}_+)$.

Definition 4.19. An asymptotically almost periodic distribution $x \in AP'^m(\mathbb{R}_+)$ one will call a generalized solution of (4.22), if $\langle x, L^*\varphi\rangle = \langle f, \varphi\rangle$ for any $\varphi \in AP^{m+1}(\mathbb{R}_+)$.

Theorem 4.5.1. *If homogeneous* (4.21) *is hyperbolic on* $\mathbb{R}_+$, *then for every asymptotically almost periodic distribution* $f \in AP'^{m+1}(\mathbb{R}_+)$ (4.22) *has at least one generalized asymptotically almost periodic solution* $\eta \in AP'^m(\mathbb{R}_+)$.

Proof. Let $\Phi : AP^{m+1}(\mathbb{R}_+) \to AP^m(\mathbb{R}_+)$ be the linear operator defined by the equation

$$(\Phi f)(t) := \int_0^{+\infty} G_A(t, \tau)f(\tau)d\tau, \tag{4.55}$$

where $G_A(t,\tau)$ is the main function of Green of (4.21). Let $\varphi \in AP^m(\mathbb{R}_+)$ and $f \in AP^{m+1}(\mathbb{R}_+)$. Then $\eta = \Phi f$ defines a regular distribution $\eta \in AP'^m(\mathbb{R}_+)$, and

$$\begin{aligned}
\langle \eta, \varphi\rangle &= M\{\langle \eta(t), \varphi(t)\rangle\} = M\Big\{\Big\langle \int_0^{+\infty} G_A(t,\tau)f(\tau)d\tau, \varphi(t)\Big\rangle\Big\} \\
&= M\Big\{\int_0^{+\infty} \langle G_A(t,\tau)f(\tau), \varphi(t)\rangle d\tau\Big\} = M\Big\{\int_0^{+\infty} \langle f(\tau), G_A^*(t,\tau)\varphi(t)\rangle d\tau\Big\} \\
&= \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_0^{+\infty} \langle f(\tau), G_A^*(t,\tau)\varphi(t)\rangle d\tau\, dt \\
&= \lim_{T\to+\infty} \frac{1}{T}\int_0^T \Big\langle f(\tau), \int_0^{+\infty} G_A^*(t,\tau)\varphi(t)dt\Big\rangle d\tau = M\Big\{\Big\langle f(\tau), \int_0^{+\infty} G_A^*(t,\tau)\varphi(t)dt\Big\rangle\Big\},
\end{aligned} \tag{4.56}$$

consequently

$$\langle \eta, \varphi\rangle = M\Big\{\Big\langle f(\tau), \int_0^{+\infty} G_A^*(t,\tau)\varphi(t)dt\Big\rangle\Big\}, \tag{4.57}$$

where $G_A^*(t,\tau)$ is the adjoint operator for $G_A(t,\tau)$.

Along with (4.21) the equation

$$\frac{dy}{dt} = -A^*(t)y. \tag{4.58}$$

is hyperbolic on $\mathbb{R}_+$ too. According to Theorem 3.3.18 the equation

$$(S^*\varphi)(t) = \int_0^{+\infty} G_A^*(t,\tau)\varphi(t)dt \tag{4.59}$$

correctly defines a mapping from $AP^m(\mathbb{R}_+)$ into $AP^{m+1}(\mathbb{R}_+)$. Let us show that the operator $S^* : AP^m(\mathbb{R}_+) \to AP^{m+1}(\mathbb{R}_+)$ is continuous. In fact, from inequality (3.4.20) from [128] it follows that

$$\int_0^{+\infty} ||G_A^*(t,\tau)||dt \le \frac{2N}{\nu} \quad (\tau \in \mathbb{R}_+), \tag{4.60}$$

where N, ν are the constants of hyperbolicity of (4.58). From inequality (4.60) we have

$$||\Phi^*\varphi||_0 = \sup_{t\in\mathbb{R}_+} \Big| \int_0^{+\infty} G_A^*(t,\tau)\varphi(t)dt\Big| \le \int_0^{+\infty} ||G_A^*(t,\tau)||dt \cdot \|\varphi\|_0 \le \frac{2N}{\nu}\|\varphi\|_m. \tag{4.61}$$

So, $\Phi^*\varphi$ is a solution of the equation

$$\frac{dy}{dt} = -A^*(t)y + \varphi(t) \tag{4.62}$$

and, consequently, $\Phi^*(AP^m(\mathbb{R}_+)) \subseteq AP^{m+1}(\mathbb{R}_+)$ and $\|\Phi^*\varphi\|_m \le c_m \cdot \|\varphi\|_m$, where c_m is some positive constant depending only on m and the matrix $A(t)$.

So, Φ^* is a linear bounded operator acting from $AP^m(\mathbb{R}_+)$ into $AP^{m+1}(\mathbb{R}_+)$. Thus we have the equality

$$\langle \Phi f, \varphi \rangle = \langle \eta, \varphi \rangle = \langle f, \Phi^* \varphi \rangle, \tag{4.63}$$

which takes place and has sense for every $\varphi \in AP^m(\mathbb{R}_+)$ and $f \in AP^{m+1}(\mathbb{R}_+)$, where $\eta = \Phi f$. From equality (4.63) follows that

$$\langle \eta, L^* \varphi \rangle = \langle \Phi f, L^* \varphi \rangle = \langle f, \Phi^* (L^* \varphi) \rangle. \tag{4.64}$$

Let now $\varphi \in AP^{m+1}(\mathbb{R}_+)$. Then

$$\begin{aligned} \Phi^* (L^* \varphi)(\tau) &= \Phi^* (\dot{\varphi}(t) + A^*(t)\varphi(t))(\tau) \\ &\dot{=} \int_0^{+\infty} G_A^*(t,\tau)\dot{\varphi(t)}dt + \int_0^{+\infty} G_A^*(t,\tau)A^*(t)\varphi(t)dt. \end{aligned} \tag{4.65}$$

From (4.65) integrating by parts we obtain

$$\int_0^{+\infty} G_A^*(t,\tau)\dot{\varphi(t)}dt = G_A^*(t,\tau)\varphi(t)|_0^{+\infty} - \int_0^{+\infty} \frac{\partial}{\partial t} G_A^*(t,\tau)\varphi(t)dt \tag{4.66}$$

$$= -G_A^*(0,\tau)\varphi(0) - \int_0^{+\infty} G_A^*(t,\tau)A^*(t)\varphi(t)dt + \varphi(t) \tag{4.67}$$

and, consequently,

$$(\Phi^* L^* \varphi)(\tau) = -G_A^*(0,\tau)\varphi(0) + \varphi(\tau). \tag{4.68}$$

Define by P a projector which projects E^n on

$$\begin{aligned} E^+ &= \left\{ x \mid x \in E^n, \sup_{t \geq 0} |U(t, -A^*)x| < +\infty \right\}, \\ \mathfrak{B} &= \overline{\{\varphi \mid \varphi \in AP^{m+1}(\mathbb{R}_+), P\varphi(0) = 0\}}, \end{aligned} \tag{4.69}$$

where by bar it is denoted the closure in $AP^{m+1}(\mathbb{R}_+)$. Then from equality (4.68) it follows that $\Phi^* L^*$ is the identical operator in $\mathfrak{B}$ and hence

$$L^* \Phi^* = Id_{AP^m(\mathbb{R}_+)}, \qquad \Phi^* L^* = Id_{\mathfrak{B}}, \tag{4.70}$$

where $Id_{AP^m(\mathbb{R}_+)}$ and $Id_{\mathfrak{B}}$ are identical operators in $AP^m(\mathbb{R}_+)$ and $\mathfrak{B}$, respectively. Now let us denote by $\overline{L} = (L^*)'$ and $\overline{\Phi} = (\Phi^*)'$ the operators adjoint to L^* and Φ^*, respectively. Then $\overline{L} \circ \overline{\Phi} = Id_{\mathfrak{B}'}$, where $\mathfrak{B}'$ is the space adjoint to $\mathfrak{B}$ and $Id_{\mathfrak{B}'}$ is the identical operator in $\mathfrak{B}'$. Note that $\overline{L}|_{AP'^{m+1}(\mathbb{R}_+)} = L$ and $\overline{\Phi}|_{AP'^{m}(\mathbb{R}_+)} = \Phi$, where $\overline{L}|_{AP'^{m+1}(\mathbb{R}_+)}$ and $\overline{\Phi}|_{AP'^{m}(\mathbb{R}_+)}$ are the restrictions of the operators $\overline{L}$ on $AP'^{m+1}(\mathbb{R}_+)$ and $\overline{\Phi}$ on $AP'^{m}(\mathbb{R}_+)$.

To finish the proof of the theorem we only need to note that

$$\langle x, L^* \varphi \rangle = \langle f, \varphi \rangle \tag{4.71}$$

for every $f \in B' \supseteq AP'^{m+1}(\mathbb{R}_+)$. In fact,

$$\langle x, L^*\varphi\rangle = \langle \overline{\Phi} f, L^*\varphi\rangle = \langle \overline{L}\,\overline{\Phi} f, \varphi\rangle = \langle f, \varphi\rangle. \tag{4.72}$$

The theorem is proved. □

4.6. Dynamical Systems of Shifts in the Spaces of Distributions and Asymptotically Almost Periodic Functions in the Sobolev Spaces

4.6.1. Adjoint Dynamical System

Let X be a vector topological space and $(X, \mathbb{R}, \pi)$ be a dynamical system on X. By X' denote the adjoint space of all linear continuous forms defined on X.

By X'_w we denote X' with the weak topology and by X'_c we denote X' with the topology of compact convergence. Then $f_j \to 0$ in X'_w if and only if $\langle f_j, x\rangle \to 0$ for every $x \in X$, and $f_j \to 0$ in X'_c if and only if for every compact set $A \subseteq X$ we have $\sup\{|\langle f_j, x\rangle| : x \in A\} \to 0$.

At the first sight the topology of compact convergence seems stronger than the weak topology. Nevertheless, for a large class of spaces (e.g., Fréchet spaces), where can be applied the theorem of Banach-Steinhaus [133, 134], these topologies are equivalent.

In this section we will consider only such spaces X for which the weak topology and the topology of compact convergence are equivalent on X'. For this it is sufficient that X would be the space of Fréchet though there exist and other spaces possessing this property.

Let $h \in \mathbb{R}$. Let us define a mapping τ_h ("h-shift") of the space X'_w into itself by the equality

$$(\tau_h f)(x) = f(\pi(x, h)) \tag{4.73}$$

for all $x \in X$ and $f \in X'_w$. It is easy to verify that the obtained family of mappings $\{\tau_h \mid h \in \mathbb{R}\}$ possesses the following properties:

$$\tau_0 = Id_{X'_w}, \tag{4.74}$$

$$\tau_{h_1} \circ \tau_{h_2} = \tau_{h_1+h_2} \quad (h_1, h_2 \in \mathbb{R}), \tag{4.75}$$

$$\tau_h : X'_w \longrightarrow X'_w \quad \text{is continuous.} \tag{4.76}$$

Define a mapping $\pi' : X'_w \times \mathbb{R} \to X'_w$ by the equality

$$\pi'(f, h) := \tau_h f \tag{4.77}$$

for every $f \in X'_w$ and $h \in \mathbb{R}$. From (4.74)–(4.75) it follows that

$$\pi'(f, 0) = f, \qquad \pi'(\pi'(f, h_1), h_2) = \pi'(f, h_1 + h_2) \tag{4.78}$$

for any $f \in X'_w$ and $h_1, h_2 \in \mathbb{R}$.

Lemma 4.20. *$(X'_w, \mathbb{R}, \pi')$ is a dynamical system.*

Proof. It is sufficient to prove the continuity of π'. Let $f_j \to f$ in X'_w and $t_j \to t$ in $\mathbb{R}$. Then for an arbitrary $x \in X$ we have $\pi'(f_j, t_j)(x) = f_j(\pi(x, t_j))$. Define a mapping $\mathcal{B} : X \times X'_w \to X$ by the equality $\mathcal{B}(x, f) = f(x)$. Let us show that this mapping is continuous. Let $x_k \to x$ and $f_k \to f$. Then

$$\begin{aligned} |f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &\leq \sup\{f_n(\overline{x}) - f(\overline{x})| : \overline{x} \in Q\} + |f(x_n) - f(x)|, \end{aligned} \tag{4.79}$$

where $Q = \{x_n\} \cup \{x\}$. Taking into account the equivalence of the weak topology and the topology of compact convergence on X' and inequality (4.6.7), we conclude that $\mathcal{B}$ is continuous. Note that

$$\pi'(f_j, t_j)(x) = f_j(\pi(x, t_j)) = \mathcal{B}(\pi(x, t_j), f_j) \longrightarrow \mathcal{B}(\pi(x, t), f). \tag{4.80}$$

Since $\pi(x, t_j) \to \pi(x, t)$ and $\mathcal{B}$ is continuous, then $\pi'(f_j, t_j)(x) \to f(\pi(x, t)) = \pi'(f, t)(x)$ for every $x \in X$, that is, $\pi'(f_j, t_j) \to \pi'(f, t)$ in X'. The lemma is proved. □

Definition 4.21. A dynamical system $(X', \mathbb{R}, \pi')$ is called an adjoint system for $(X, \mathbb{R}, \pi)$.

4.6.2. Dynamical Systems of Shifts on $\mathcal{D}$ and $\mathcal{D}'$

Recall (see Section 4.1) that by $\mathcal{D} = \mathcal{D}(\mathbb{R})$ we denoted the space of all finite and infinitely differentiable functions $\varphi : \mathbb{R} \to \mathbb{R}^n$. The space $\mathcal{D}$ with the introduced in it convergence is a locally convex vector topological space but not the space of Fréchet.

Define for every $h \in \mathbb{R}$ a mapping $\tau_h : \mathcal{D} \to \mathcal{D}$ by the following rule:

$$(\tau_h \varphi)(x) := \varphi(x + h) \tag{4.81}$$

for all $\varphi \in \mathcal{D}$ and $x \in \mathbb{R}$. Conditions (4.74)–(4.76) are verified easily. With the help of the family of mappings $\{\tau_h\}_{h \in \mathbb{R}}$, we define a mapping $\sigma : \mathbb{R} \times \mathcal{D} \mapsto \mathcal{D}$ by formula (4.77). It is easy to see that σ satisfies identities (4.78). Let us show that $\sigma : \mathbb{R} \times \mathcal{D} \mapsto \mathcal{D}$ is continuous.

Let $\varphi_k \to \varphi$ in $\mathcal{D}$ and $h_k \to h$ in $\mathbb{R}$. Since $\varphi_k \to \varphi$ and $h_k \to h$, then there exists a compact $K \subset \mathbb{R}$ such that

$$-h_k + \operatorname{supp} \varphi_k \subseteq K \quad (k = 1, 2, \ldots) \tag{4.82}$$

and, consequently, $\operatorname{supp} \tau_{h_k} \varphi_k \subseteq K \quad (k = 1, 2, \ldots)$. Let us show that the sequence $\{\sigma(h_k, \varphi_k)\}$ converges to $\sigma(h, \varphi)$ in $\mathcal{D}$. In fact, let $x \in K$. Then

$$\begin{aligned} &|D^j \sigma(h_k, \varphi_k)(x) - D^j \sigma(h, \varphi)(x)| \\ &\quad = |D^j \varphi_k(x + h_k) - D^j \varphi(x + h)| \\ &\quad \leq |D^j \varphi_k(x + h_k) - D^j \varphi(x + h_k)| + |D^j \varphi_k(x + h_k) - D^j \varphi(x + h)| \\ &\quad \leq \max_{y \in K'} |D^j \varphi_k(y) - D^j \varphi(y)| + \max_{z \in K} |D^j \varphi(z + h_k) - D^j \varphi(z + h)|, \end{aligned} \tag{4.83}$$

where K' is a compact from $\mathbb{R}$ such that $\operatorname{supp}\varphi_k \subseteq K'$. From inequality (4.83) we can easily see that to finish the proof of the convergence of $\{\sigma(h_k,\varphi_k)\}$ to $\sigma(h,\varphi)$ in $\mathcal{D}$ it is easy to show that for every $j \in \mathbb{Z}_+$ there takes place the equality

$$\lim_{k\to+\infty}\max_{x\in K}\left|D^j\varphi(x+h_k) - D^j\varphi(x+h)\right| = 0. \tag{4.84}$$

Suppose the contrary, that is, there exist $j_0 \in \mathbb{Z}_+$, $\varepsilon_0 > 0$, and $\{x_k\} \subseteq K$ such that

$$\left|D^{j_0}\varphi(x_k+h_k) - D^{j_0}\varphi(x_k+h)\right| \geq \varepsilon_0. \tag{4.85}$$

Since K is a compact, then without loss of generality we can consider that $x_k \to x_0$ and passing to the limit in (4.85) as $k \to +\infty$ we obtain that

$$0 = \left|D^{j_0}\varphi(x_0+h) - D^{j_0}\varphi(x_0+h)\right| \geq \varepsilon_0. \tag{4.86}$$

The last inequality contradicts to the choice of ε_0. Hence, the needed assertion is proved. So, the mapping $\sigma : \mathbb{R}\times\mathcal{D} \mapsto \mathcal{D}$ is continuous and consequently the triplet $(\mathcal{D},\mathbb{R},\sigma)$ is a dynamical system of shifts on $\mathcal{D}$.

Denote by $\mathcal{D}'$ the set of all linear continuous forms on $\mathcal{D}$ endowed with the weak topology, that is, $f_k \to f$ in $\mathcal{D}'$ if and only if $(f_k,\varphi) \to (f,\varphi)$ for every $\varphi \in \mathcal{D}$. Defined in this way topology on $\mathcal{D}'$ turns it into locally convex vector topological space. As we mentioned above, the space $\mathcal{D}$ is not a space of Fréchet, nevertheless the weak and compact convergence on $\mathcal{D}'$ coincide. In fact, let $f_k \to f$ in the weak topology and M is an arbitrary compact set from $\mathcal{D}$. Then according to [134, 135], there exists a compact $K \subset \mathbb{R}$ such that $M \subset \mathcal{D}_k \subset \mathcal{D}$, where $\mathcal{D}_k$ is the set of all functions from $\mathcal{D}$, the supports of which are in K. Since $(f_k,\varphi) \to (f,\varphi)$ for every $\varphi \in \mathcal{D}$, then in particular $(f_k,\varphi) \to (f,\varphi)$ also as $\varphi \in \mathcal{D}_k$. Therefore, $\{f_k\}$ is weakly convergent on $\mathcal{D}_k$. But the space $\mathcal{D}_k$ is a space of Fréchet and for these spaces the weak topology and the topology of the compact convergence are equivalent. That is why $f_k \to f$ is uniformly on M.

Resuming all the said above we conclude that on the space $\mathcal{D}'$ there is defined a dynamical system $(\mathcal{D}',\mathbb{R},\sigma')$ that is adjoint to $(\mathcal{D},\mathbb{R},\sigma)$.

4.6.3. Dynamical Systems on the Local Spaces

Definition 4.22. The space $\mathcal{F} \subseteq \mathcal{D}'$ is called [137] semilocal, if $\varphi u \in \mathcal{F}$ for every $u \in \mathcal{F}$ and $\varphi \in C_0^\infty = C_0^\infty(\mathbb{R})$. If $\mathcal{F}$ contains every distribution $u \in \mathcal{D}'$, for which $\varphi u \in \mathcal{F}$ for every $\varphi \in C_0^\infty$, then $\mathcal{F}$ is called local.

The least local space containing $\mathcal{F}$ we denote by $\mathcal{F}_{\text{loc}}$. $\mathcal{F}_{\text{loc}}$ is the space defined by the following rule [138]:

$$\mathcal{F}_{\text{loc}} := \{u \mid u \in \mathcal{D}',\ \varphi u \in \mathcal{F} \text{ for every } \varphi \in C_0^\infty\}. \tag{4.87}$$

Let us denote by $\mathcal{F}^c$ the set of al $u \in \mathcal{F}$ with the compact support [135]. If $\mathcal{F}$ is semilocal, then according to [138]:

$$\mathcal{F}^c = \mathcal{F}_{\text{loc}} \cap \mathcal{E}', \qquad \mathcal{F}_{\text{loc}} = (\mathcal{F}^c)_{\text{loc}}, \tag{4.88}$$

where $\mathcal{E}' := \mathcal{D}'^c$, here $\mathcal{D}'^c$ is the set of all $u \in \mathcal{D}'$ with the compact support.

Let $\mathcal{F} \subseteq \mathcal{D}'$ be semilocal normed subspace of the space $\mathcal{D}'$ with the norm $\|\cdot\|_{\mathcal{F}}$. Further we will omit the index $\mathcal{F}$, if it is clear what norm is meant. On the space $\mathcal{F}_{\text{loc}}$ we can define a topology τ by a family of seminorms as following. If $\varphi \in C_0^\infty$, then the mapping $p_\varphi : \mathcal{F}_{\text{loc}} \to \mathbb{R}_+$ defined by the equality

$$p_\varphi(u) = \|\varphi u\|, \tag{4.89}$$

gives some seminorm on $\mathcal{F}_{\text{loc}}$. The family of seminorms $\mathcal{P} = \{p_\varphi \mid \varphi \in C_0^\infty\}$ defines on $\mathcal{F}_{\text{loc}}$ some topology.

Lemma 4.23. *Let $\mathcal{F} \subseteq \mathcal{D}'$ be a semilocal normed subspace of the space $\mathcal{D}'$ such that multiplication by a function from C_0^∞ is continuous (i.e., for every $\varphi \in C_0^\infty$ there exists a positive constant $M(\varphi) > 0$ such that $\|u\varphi\| \le M(\varphi)\|u\|$ for every $u \in \mathcal{F}$). Then the topology τ generated by the family of seminorms $\mathcal{P}$ on $\mathcal{F}_{\text{loc}}$ is metrizable.*

Proof. Denote by $\varphi_k \in C_0^\infty$ a function satisfying the following two conditions:

$$[-k,k] \subseteq \operatorname{supp}\varphi_k \subseteq [-(k+1),k+1], \tag{4.90}$$

$$\varphi_k(x) = 1 \quad \text{for all } x \in [-k,k]. \tag{4.91}$$

It is well known [135] that such functions exist. Let us consider a countable family of seminorms $\mathcal{P}' = \{p_k = p_{\varphi_k} \mid k = 1,2,\ldots\}$. Let us show that the family of seminorms $\mathcal{P}'$ is separating on $\mathcal{F}_{\text{loc}}$ [133, 134]. In fact, let $u \in \mathcal{F}_{\text{loc}}$ be not equal to zero. Then there exists $\varphi \in C_0^\infty$ such that $p_\varphi(u) = \|u\varphi\| > 0$. Since $\varphi \in C_0^\infty$, then there exists a number k such that $\operatorname{supp}\varphi \subseteq [-k,k]$ and, consequently, $\varphi\varphi_k = \varphi$. Then we have

$$0 < \|u\varphi\| = ||u\varphi\varphi_k|| \le M(\varphi)||u\varphi_k|| = M(\varphi)p_k(\varphi). \tag{4.92}$$

So, $p_k(\varphi) > 0$. Hence, the family of seminorms $\{p_k\}$ is separating and the formula

$$d(u,v) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(u-v)}{1+p_k(u-v)} \tag{4.93}$$

defines an invariant metric on $\mathcal{F}_{\text{loc}}$ which is compatible with the topology τ [133, 134]. The lemma is proved. □

Remark 4.24. $\mathcal{F}_{\text{loc}}$ with the metric (4.93) is a complete metric space if $\mathcal{F}$ is a Banach space.

Lemma 4.25. *Let $(\mathcal{D}',\mathbb{R},\sigma')$ be a dynamical system of shifts on $\mathcal{D}'$. If the restriction σ' on $\mathcal{F}\times\mathbb{R}$ is continuous in $\mathcal{F}\times\mathbb{R}$, where $\mathcal{F}$ is a subspace of $\mathcal{D}'$ satisfying the conditions of Lemma 4.23, then the restriction σ' on $\mathcal{F}_{\text{loc}}\times\mathbb{R}$ is continuous in the topology $\mathcal{F}_{\text{loc}}\times\mathbb{R}$.*

Proof. Let $u_p \to u$ in $\mathcal{F}_{\text{loc}}$ and $h_p \to h$. Let us show that $\sigma'(u_p,h_p) \to \sigma'(u,h)$ in $\mathcal{F}_{\text{loc}}$. Since $h_p \to h$, then there exists $h_0 > 0$ such that $|h_p| \le h_0$ for all $p = 1,2,\ldots$. Let $A \subset \mathbb{R}$. Denote by $B[A,h_0] = \{x+y \mid x \in A,\ |y| \le h_0\}$. Let us estimate

$$p_k(\sigma'(u_p,h_p) - \sigma'(u,h)) = ||\sigma'(u_p,h_p) - \sigma'(u,h)\varphi_k||, \tag{4.94}$$

for this aim we choose $\varphi \in C_0^\infty$ so that $\varphi(x) = 1$ for all $x \in B[[-k,k],h_0]$. Then

$$\varphi(x+\tau)\varphi_k(x) = \varphi_k(x) \quad \forall x \in [-k,k],\ |\tau| \le h_0. \tag{4.95}$$

So,

$$\begin{aligned}
&p_k(\sigma'(u_p,h_p) - \sigma'(u,h)) \\
&\quad = ||[\sigma'(u_p,h_p) - \sigma'(u,h)]\varphi_k|| = ||\sigma'(u_p,h_p)\varphi_k - \sigma'(u,h)\varphi_k|| \\
&\quad = ||\sigma'(u_p,h_p)\sigma(\varphi,h_p)\varphi_k - \sigma'(u,h)\sigma(\varphi,h)\varphi_k|| \\
&\quad = ||\sigma'(u_p\varphi,h_p)\varphi_k - \sigma'(u\varphi,h)\varphi_k|| = ||[\sigma'(u_p\varphi,h_p) - \sigma'(u\varphi,h)]\varphi_k|| \\
&\quad \le M(\varphi_k)||\sigma'(u_p\varphi,h_p) - \sigma'(u\varphi,h)||.
\end{aligned} \tag{4.96}$$

Since $\varphi \in C_0^\infty$, then $u_p\varphi \to u\varphi$ in $\mathcal{F}$ and, consequently,

$$\lim_{p\to+\infty} ||\sigma'(u_p\varphi,h_p) - \sigma'(u\varphi,h)|| = 0. \tag{4.97}$$

From (4.96) and (4.97) it follows that

$$\lim_{p\to+\infty} p_k(\sigma'(u_p,h_p) - \sigma'(u,h)) = 0 \tag{4.98}$$

for every $k = 1,2,\ldots$. The lemma is proved. □

Corollary 4.26. *The triplet* $(\mathcal{F}_{\text{loc}}, \mathbb{R}, \sigma')$ *is a dynamical system of shifts on* $\mathcal{F}_{\text{loc}}$, *where* $\mathcal{F}$ *is a semilocal normed subspace of* $\mathcal{D}'$ *in which multiplication by functions from* C_0^∞ *is continuous.*

4.6.4. Dynamical Systems of Shifts and Asymptotically Almost Periodic Functions in the Sobolev Spaces H^s

Let $k : \mathbb{R} \to \mathbb{R}$ be a continuous positive function satisfying to the inequality

$$k(\xi)k^{-1}(\eta) \le C(1+|\xi-\eta|)^l \quad (\xi,\eta \in \mathbb{R}) \tag{4.99}$$

for some constant C and l depending only on the function k. Denote by $L^{k,p}$ the set of all measurable in $\mathbb{R}$ functions $u : \mathbb{R} \to \mathbb{R}^n$ for which the integral

$$\|u\|^p = \int |u(\xi)|^p k^p(\xi)d\xi, \tag{4.100}$$

is finite, where $1 \le p \le +\infty$. If $k(\xi) = 1$, then the space $L^{k,p}$ coincides with the space L^p. The spaces $L^{k,p}$ and L^p are isometrically isomorphic [137, 138]. In particular, $L^{k,p}$ is a reflexive Banach space.

Let $H^{k,p}$ be the family of those distributions $u \in \mathcal{D}'$, the Fourier-image of which $\hat{u} = \mathcal{F}u$ belongs to the space $L^{k,p}$. The topology on $H^{k,p}$ is given with the help od the norm

$$\|u\| = \left(\int |\hat{u}(\xi)|^p k^p(\xi)d\xi\right)^{1/p}. \tag{4.101}$$

It is known [137, 138] that the operator of Fourier $\mathcal{F}$ establishes an isometric isomorphism between $H^{k,p}$ and $L^{k,p}$. Therefore $H^{k,p}$ is a reflexive Banach space.

Let $(\mathcal{D}', \mathbb{R}, \sigma')$ be a dynamical system of shifts on $\mathcal{D}'$. Let us show that the restriction on $\mathbb{R} \times H^{k,p}$ of the mapping $\sigma' : \mathbb{R} \times \mathcal{D} \mapsto \mathcal{D}'$ is continuous in the topology $\mathbb{R} \times H^{k,p}$. In fact, if $u_k \to u$ in $H^{k,p}$ and $h_r \to h$ in $\mathbb{R}$, then

$$||\sigma'(h_r, u_r) - \sigma'(h, u)|| \le ||\sigma'(h_r, u_r) - \sigma(h_r, u)|| + ||\sigma'(h_r, u) - \sigma'(h, u)||. \quad (4.102)$$

There is known [137, page 18] that the space $H^{k,p}$ is invariant with respect to the shifts τ_a ($a \in \mathbb{R}$), and for $u \in H^{k,p}$ there take place the equalities

$$\|u\| = ||\tau_a u||, \qquad \lim_{|a| \to 0} ||\tau_a u - u|| = 0. \quad (4.103)$$

From (4.102) and (4.103), it follows that $\sigma'(h_r, u_r) \to \sigma'(h, u)$ in $H^{k,p}$ as $k \to \infty$.

Corollary 4.27. *The triplet $(H^{k,p}, \mathbb{R}, \sigma')$ is a dynamical systems of shifts on $H^{k,p}$.*

So, in the space $\mathcal{D}'$ we take a Banach subspace $H^{k,p}$ such that the restriction σ' on $\mathbb{R} \times H^{k,p}$ is continuous in the topology $\mathbb{R} \times H^{k,p}$. From [137, 138] it follows that the subspace $H^{k,p}$ is semilocal and the operation of multiplication by functions from C_0^∞ is continuous. According to Lemma 4.23 on the subspace $H^{k,p}_{\text{loc}}$ given by formula (4.87), the family of seminorms (4.89) defines a metrizable topology.

Corollary 4.28. *The triplet $(H^{k,p}_{\text{loc}}, \mathbb{R}, \sigma')$ is a dynamical system of shifts on $H^{k,p}_{\text{loc}}$.*

Proof. The formulated statement follows from the above said and Lemma 4.25. □

By H^s and H^s_{loc} we denote spaces $H^{k,p}$ and $H^{k,p}_{\text{loc}}$, respectively, in the case when $k(\xi) = (1 + |\xi|^2)^s$ and $p = 2$. From Corollaries 4.27 and 4.28 it follows that on the spaces H^s and H^s_{loc} there are defined the dynamical systems of shifts $(H^s, \mathbb{R}, \sigma')$ and $(H^s_{\text{loc}}, \mathbb{R}, \sigma')$, respectively.

As well as for continuous functions, the dynamical systems $(H^{k,p}_{\text{loc}}, \mathbb{R}, \sigma')$ and $(H^s_{\text{loc}}, \mathbb{R}, \sigma')$ give a useful means of the study of general properties of functions from $H^{k,p}_{\text{loc}}$ involving the general theory of dynamical systems.

For example, a function $u \in H^{k,p}_{\text{loc}}$ we will call almost periodic (resp., asymptotically almost periodic), if the motion $\sigma'(\cdot, u)$ generated by the function u in the dynamical system $(H^{k,p}_{\text{loc}}, \mathbb{R}, \sigma')$ is almost periodic (resp., asymptotically almost periodic).

4.7. Weakly Asymptotically Almost Periodic Functions

Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{R}_+$. Denote by $C_b(\mathbb{T}, E^n)$ the Banach space of all continuous and bounded functions $f : \mathbb{T} \to E^n$ endowed with the norm $\|f\| = \sup\{|f(t)| : t \in \mathbb{T}\}$. Note that the space $C_b(\mathbb{T}, E^n)$ is isomorphic to the space $(C_b(\mathbb{T}, E))^n := C_b(\mathbb{T}, E) \times C_b(\mathbb{T}, E) \times \cdots \times C_b(\mathbb{T}, E)$.

Denote by $\tau_h f$ the shift of the function $f \in (C_b(\mathbb{T},E))^n$, that is, $(\tau_h f)(x) := f(x+h)$, $(C_b^*(\mathbb{T},E))^n$ is the adjoint space for $(C_b(\mathbb{T},E))^n$. If $\varphi \in (C_b^*(\mathbb{T},E))^n$ and $f \in (C_b(\mathbb{T},E))^n$, then $\langle \varphi, f\rangle \in E^n$. By the sign $\rightharpoonup$ we will denote the weak convergence of sequences in $(C_b(\mathbb{T},E))^n$.

Definition 4.29. A function $f \in (C_b(\mathbb{R}_+,E^n))^n$ is called weakly asymptotically almost periodic, if the set of shifts $\{\tau_h f : h \in \mathbb{R}_+\}$ forms a relatively compact set in the weak topology $(C_b(\mathbb{R}_+,E^n))^n$.

The set of all weakly asymptotically almost periodic functions we denote by $\mathcal{A}_w$.

Taking into account the equivalence of the properties of compactness and countable compactness (see, i.e., [139, Theorem 1.2]), we obtain the following statement.

Lemma 4.30. *$f \in (C_b(\mathbb{R}_+,E^n))^n$ is a weakly asymptotically almost periodic function if and only if for every sequence $\{h_k\} \subset \mathbb{R}_+$ there exist a subsequence $\{h_{k_m}\}$ and a function $g \in (C_b(\mathbb{R}_+,E))^n$ such that $\tau_{h_{k_m}} f \rightharpoonup g$, that is, $\langle \varphi, \tau_{h_{k_m}}\rangle \to \langle \varphi, g\rangle$ for every function $\varphi \in (C_b^*(\mathbb{R}_+,E))^n$.*

Lemma 4.31. *If $f = \lim_{k\to+\infty} f_k$ $g = \lim_{k\to+\infty} g_k$ in the weak topology $C_b(\mathbb{R}_+,E)$, then $fg = \lim_{k\to+\infty} f_k g_k$ in the weak topology $C_b(\mathbb{R}_+,E)$.*

Proof. According to Theorem of Gelfand-Neimark [140] the space $C_b(\mathbb{R}_+,E)$ is isometrically isomorphic to the ring $C(\Omega)$ of all complex-valued functions on the compact Hausdorff space Ω (where Ω is the space of maximal ideals).

Taking into account all the above said, without loss of generality we can consider that $f_k, g_k \in C(\Omega)$. Since the weak convergence of $\{f_k\}$ in $C(\Omega)$ is equivalent to its boundedness and point convergence, then $f_k g_k \rightharpoonup fg$. The lemma is proved. □

Theorem 4.7.1. *The set $\mathcal{A}_w$ is a closed subalgebra of $(C_b(\mathbb{R}_+,E))^n$ and is invariant with respect to shifts.*

Proof. From the definition it follows that if $f \in \mathcal{A}_w$, then $\tau_h f \in \mathcal{A}_w$ $(h \in \mathbb{R}_+)$. Since $\mathcal{A}_w$ is a convex subset of $(C_b(\mathbb{R}_+,E))^n$, then according to [139, Theorem 1.1] for proving the closure of $\mathcal{A}_w$ in the weak topology it is sufficient to prove its closure in the topology $(C_b(\mathbb{R}_+,E))^n$. Suppose that $f = \lim_{k\to+\infty} f_k$ $(\{f_k\} \subset \mathcal{A}_w)$, that is, $\|f_k - f\| \to 0$ for $k \to +\infty$, where $\|\cdot\|$ is the norm in $(C(\mathbb{R}_+,E))^n$, and let $\{h_m\} \subset \mathbb{R}_+$. Then there exists a subsequence $\{h_{m_p}\} \subset \{h_m\}$ and elements g_m such that $\lim_{p\to+\infty} \tau_{h_{m_p}} f_k = g_k$ in the weak topology $(k = 1, 2, \dots)$. If we show that there exists $g = \lim_{k\to+\infty} g_k$, then it will imply that $g = \lim_{p\to+\infty} \tau_{h_{m_p}} f$ in the weak topology and, consequently, $f \in \mathcal{A}_w$. From the theorem of Hahn-Banach it follows that

$$\begin{aligned} ||g_r - g_s|| &= \sup\{|\langle \varphi, g_r - g_s\rangle| : \|\varphi\| \le 1\} \\ &= \sup_{\|\varphi\|\le 1} \lim_{p\to+\infty} |\langle \varphi, \tau_{h_{mp}}(f_r - f_s)\rangle| \le ||f_r - f_s||. \end{aligned} \tag{4.104}$$

Then $\{g_k\}$ is fundamental and, consequently, there exists $\lim_{k\to+\infty} g_k = g$.

To finish the proof of the theorem let us show that if $f^1, f^2 \in \mathcal{A}_w$, then $f^1 \cdot f^2 \in \mathcal{A}_w$. Let $f^1, f^2 \in \mathcal{A}_w$ and $\{h_m\} \subset \mathbb{R}_+$. We choose a subsequence $\{h_{m_p}\}$ and elements F^1, F^2 in $C_b(\mathbb{R}_+, E)$ such that

$$\lim_{p\to+\infty} \tau_{h_{m_p}} f^1 = F^1, \qquad \lim_{p\to+\infty} \tau_{h_{m_p}} f^2 = F^2 \tag{4.105}$$

in the weak topology. To conclude that

$$\lim_{p\to+\infty} \tau_{h_{m_p}} f^1 \cdot \tau_{h_{m_p}} f^2 = F^1 \cdot F^2 \tag{4.106}$$

in the weak topology it is enough to refer to Lemma 4.31. The theorem is proved. □

Corollary 4.32. *$\mathcal{A}_w$ equipped with the norm $\|f\| := \sup\{|f(t)| : t \in \mathbb{R}_+\}$ is a Banach space.*

Let $M \subset E^n$ be an open set. Denote by $C_b(\mathbb{R}_+ \times M; E^n) = (C_b(\mathbb{R}_+ \times M; E))^n$ the set of all continuous functions defined on $\mathbb{R}_+ \times M$ with values in E^n and bounded on every set $\mathbb{R}_+ \times K$, where $K \subset M$ is a compact set. For the function $f \in (C_b(\mathbb{R}_+ \times M; E))^n$ by $\tau_h f$ we denote the shift of the function f with respect to t on h, that is, $(\tau_h f)(t,p) := f(t+h,p)$, and $\tau_h f := f^h$.

Definition 4.33. A function $f \in (C_b(\mathbb{R}_+ \times M; E))^n$ one will call weakly asymptotically almost periodic with respect to t uniformly with respect to $p \in M$, if for every subsequence $\{t_k\} \subset \mathbb{R}_+$ there exist a subsequence $\{t_{k_m}\}$ and a function $g \in C_b(\mathbb{R}_+ \times M; E^n)$ such that $\langle \varphi, f^{(t_{k_m})}(\cdot,p)\rangle \to \langle \varphi, g(\cdot,p)\rangle$ as $m \to +\infty$ for every $\varphi \in (C_b^*(\mathbb{R}, E))^n$ uniformly with respect to p on every compact subset $K \subset M$.

Lemma 4.34. *The function $f \in (C_b(\mathbb{R}_+ \times M; E))^n$ is weakly asymptotically almost periodic with respect to t uniformly with respect $p \in M$ if and only if $f(\cdot,p) \in (C_b(\mathbb{R}_+, E))^n$ is weakly asymptotically almost periodic for every $p \in M$ and the mapping $M \ni p \to f(\cdot,p) \in (C_b(\mathbb{R}_+, E))^n$ is continuous.*

Proof. Necessity. Let f be weakly asymptotically almost periodic with respect to t uniformly with respect to $p \in M$. Then from the respective definition it follows that $f(\cdot,p)$ is weakly asymptotically almost periodic for every $p \in M$. Suppose that the mapping $p \to f(\cdot,p)$ is not continuous in some point $p_0 \in M$. Then there exist $\varepsilon_0 > 0$, $\{t_k\} \subset \mathbb{R}_+$, $\{p_k\} \subset K \subset M$ such that K is a compact set, $p_k \to p_0$, and

$$|f(t_k, p_k) - f(t_k, p_0)| \geq 4\varepsilon_0 \tag{4.107}$$

for all $k \in \mathbb{N}$. Since f is weakly asymptotically almost periodic with respect to $t \in \mathbb{R}_+$ uniformly with respect to $p \in M$, then there exist $\{t_{k_m}\}$ and $g \in (C(\mathbb{R}_+ \times M; E))^n$ such that $\tau_{t_{k_m}} f \rightharpoonup g$. Then for every $\varphi \in (C_b^*(\mathbb{R}_+, E))^n$ and compact set K we will have

$$|\langle \varphi, \tau_{t_{k_m}} f(\cdot,p)\rangle - \langle \varphi, g(\cdot,p)\rangle| < \varepsilon_0 \tag{4.108}$$

for $n > n_1$ and any $p \in K$. For $\varphi = \delta_0$ (δ_0 is the measure of Dirac concentrated at the point 0) we have

$$|\tau_{t_{k_m}} f(0, p_{k_m}) - g(0, p_{k_m})| < \varepsilon, \tag{4.109}$$

$$|\tau_{t_{k_m}} f(0, p_0) - g(0, p_0)| < \varepsilon \quad (n > n_1). \tag{4.110}$$

Further, the continuity of g at the point $(0, p_0)$ imply that there is $n_2 > n_1$ such that

$$|g(0, p_{k_m}) - g(0, p_0)| < \varepsilon \quad (m > n_2 > n_1). \tag{4.111}$$

From inequalities (4.109)–(4.111), we get that for every $m > n_2$

$$\begin{aligned} &|\tau_{t_{k_m}} f(0, p_{k_m}) - \tau_{t_{k_m}} f(0, p_0)| \\ &\quad \leq |\tau_{t_{k_m}} f(0, p_{k_m}) - g(0, p_{k_m})| + |\tau_{t_{k_m}} f(0, p_0) - g(0, p_0)| \\ &\quad\quad + |g(0, p_{k_m}) - g(0, p_0)| < 3\varepsilon, \end{aligned} \tag{4.112}$$

and that contradicts to inequality (4.107).

Sufficiency. Let $\{t_k\}$ be a sequence of real numbers. Let us take a countable dense everywhere set $\{p_i\}$ from M. As $f(\cdot, p_i) \in \mathcal{A}_w$, we can find a subsequence $\{t_{k_m}\}$ such that $\tau_{t_{k_m}} f(\cdot, p_i) \rightharpoonup g(\cdot, p_i)$ for every $i \in \mathbb{N}$. By [139, Theorem 1.1] from the convex envelop of the sequence $\{\tau_{t_{k_m}} f\}$ we can construct a subsequence $\{H_m\}$ such that $\{H_m(\cdot, p)\}$ converges uniformly on $\mathbb{R}_+$ to $g(\cdot, p)$ uniformly with respect to $p \in K$. From this and from the continuity of $p \to f(\cdot, p)$ we obtain that $\{H_m(\cdot, p)\}$ satisfies the criterion of Cauchy uniformly with respect to p on the compact subset $K \subset M$. So, there exists $g(\cdot, p) \in (C_b(\mathbb{R}_+ \times M; E))^n$ $(p \in M)$ such that for every compact $K \subset M$ we have

$$\lim_{m \to +\infty} \sup_{p \in K} |H_m(\cdot, p) - g(\cdot, p)| = 0. \tag{4.113}$$

Consequently, $p \to g(\cdot, p)$ is continuous. For every fixed $\varphi \in (C^*(\mathbb{R}_+, E))^n$ we have

$$\langle \varphi, \tau_{t_{k_m}} f(\cdot, p) \rangle \longrightarrow \langle \varphi, g(\cdot, p) \rangle \tag{4.114}$$

uniformly with respect to $p \in K \subset M$. The lemma is proved. □

Lemma 4.35. *If the mapping $p \to f(\cdot, p)$ $(p \in M)$ is continuous, then for every compact subset $K \subset M$ and $\varepsilon > 0$ there exist $p_1, p_2, \ldots, p_m \in K$ and polynomials Q_i on E^n $(i = \overline{1, m})$ such that*

$$\left| f(t, p) - \sum_{i=1}^{m} f(t, p_i) Q_i(p) \right| < \varepsilon \tag{4.115}$$

for every $t \in \mathbb{R}_+$ and $p \in K$.

Proof. Let K be a compact subset from M and $\varepsilon > 0$. The set $\{f(\cdot, p) : p \in K\}$ is compact and therefore there exists a finite $\varepsilon/2$ net $\{f(\cdot, p_i) \mid i = \overline{1, m}\}$. Assume $U_i = \{p : \|f(\cdot, p) - f(\cdot, p_i)\| < \varepsilon/2\}$. Let g_i $(i = \overline{1, m})$ be the elements of the decomposition of the unit for the covering $\{U_i\}$ from K. Then

$$\left| f(t, p) - \sum_{i=1}^{m} f(t, p_i) g_i(p) \right| < \frac{\varepsilon}{2}. \tag{4.116}$$

Now if we approximate g_i on K with the polynomial Q_i $(i = \overline{1, m})$, we obtain the statement of the lemma. The lemma is proved. □

Lemma 4.36. *If $f \in (C_b(\mathbb{R}_+ \times M; E))^n$ is weakly asymptotically almost periodic with respect to t uniformly with respect to $p \in M$, $y \in \mathcal{A}_{wk}$ and $Q := \overline{y(\mathbb{R}_+)} \subset M$, then $W \in \mathcal{A}_w$, where $W(t) := f(t, y(t))$ for all $t \in \mathbb{R}$.*

Proof. Let $\varepsilon > 0$. By Lemma 4.35 there is $g \in (C_b(\mathbb{R}_+, E))^n$ such that $|W(t) - g(t)| < \varepsilon$ for all $t \in \mathbb{R}_+$, and

$$g(t) = \sum_{i=1}^{m} f(t, p_i) \cdot Q_i(y(t)). \tag{4.117}$$

Since $f(\cdot, p_i) \in \mathcal{A}_w$, Q_i are some polynomials and $\mathcal{A}_w$ is a subalgebra of the algebra $(C_b(\mathbb{R}_+, E))^n$, then $g \in \mathcal{A}_w$. As ε is arbitrary and $\mathcal{A}_w$ is a closed set, then W is weakly asymptotically almost periodic. The lemma is proved. □

Let $f \in (C_b(\mathbb{R}_+ \times M, E))^n$. By $H^+(f)$ we denote the set of all weakly limit points $\{\tau_n f : h \in \mathbb{R}_+\}$, that is,

$$H^+(f) := \{g \mid g \in (C_b(\mathbb{R}_+ \times M, E))^n, \exists \{t_k\} \subset \mathbb{R}_+, \tau_{t_k} f \rightharpoonup g\}. \tag{4.118}$$

Let us establish some additional properties of weakly asymptotically almost periodic functions.

Lemma 4.37. *If $f \in (C_b(\mathbb{R}_+ \times M, E))^n$ is weakly asymptotically almost periodic. With respect to t uniformly with respect to $p \in M$, then all functions in $H^+(f)$ are weakly asymptotically almost periodic with respect to t uniformly with respect to $p \in M$ too.*

Proof. Let $g \in H^+(f)$, $\tau_{t_k} f \rightharpoonup g$. By Lemma 4.34 and Theorem 4.7.1 we have $g(\cdot, p) \in \mathcal{A}_w$ for every $p \in M$. For fixed $p, q \in M$ we will have

$$|g(t, p) - g(t, q)| = \lim_{k \to +\infty} |\tau_{t_k} f(t, p) - \tau_{t_k} f(t, q)| \leq ||f(\cdot, p) - f(\cdot, q)|| \tag{4.119}$$

for every $t \in \mathbb{R}_+$. Hence, the mapping $p \to g(\cdot, p)$ is continuous and according to Lemma 4.34 $g \in H^+(f)$ is weakly asymptotically almost periodic with respect to t uniformly with respect to $p \in M$. The lemma is proved. □

Lemma 4.38. *If $f \in (C_b(\mathbb{R}_+ \times M, E))^n$ is weakly asymptotically almost periodic with respect to t uniformly with respect to $p \in M$, then $H^+(g) \subseteq H^+(f)$ for every $g \in H^+(f)$.*

Proof. Let $g \in H^+(f)$. Then there exists $\{t_k\} \subset \mathbb{R}_+$ such that $\tau_{t_k} f \rightharpoonup g$. Let us show that for every $h \in \mathbb{R}_+$, $\tau_{t_k+h} f \rightharpoonup \tau_h g$. In fact, $\{\tau_{t_k+h} f\}$ is a relatively compact sequence in the weak topology, since $f \in \mathcal{A}_w$. We will show that $\{\tau_{t_k+h} f\}$ weakly converges. For this it is sufficient to show that it contains a single limiting point. Suppose that it is not so. Then there exist $g^1, g^2 \in H^+(f)$ and $\{t_k^i + h\} \subset \{t_k + h\}$ such that $\tau_{t_k^i+h} f \rightharpoonup g^i$ $(i = 1, 2)$, and, consequently,

$$g^i(t,p) = \lim_{k\to+\infty} \tau_{t_k^i+h} f(t,p). \tag{4.120}$$

Since

$$\tau_h g(t,p) = \lim_{k\to+\infty} \tau_{t_k+h} f(t,p) \tag{4.121}$$

for every $t \in \mathbb{R}_+$ and $p \in M$, $\{t_k^i + h\} \subseteq \{t_k + h\}$, then we get $g^1(t,p) = g^2(t,p) = g^{(h)}(t,p)$. So, $\tau_{t_k+h} f \rightharpoonup \tau_h g$ and therefore $H^+(g) = \overline{\{g^{(h)} \mid h \in \mathbb{R}_+\}} \subseteq H^+(f)$, as $\tau_h g \in H^+(f)$ for all $h \in \mathbb{R}_+$ and $H^+(f)$ is closed. From this it follows that $H^+(g)$ is a compact set in the weak topology. The lemma is proved. □

Corollary 4.39. *The convergence $\{\psi_k\} \to y$ is weak in $(C_b(\mathbb{R}_+, E))^n$ if and only if $\{\psi_k\}$ is bounded and $\langle \varphi, \psi_k \rangle \to \langle \varphi, y \rangle$ for every $\varphi \in (C_b^*(\mathbb{R}_+, E))^n$, $\varphi = (\beta, \beta, \dots, \beta)$, where β is a linear multiplicative functional.*

Proof. This statement follows from the theory of the maximal ideals of Gelfand-Neimark and from the specific character of the weak convergence in $C(\Omega)$, where Ω is a compact Hausdorff space [140]. □

Following [141], we denote by $\widehat{f}$ the function defined by the next rule: $\widehat{f}(s,p) = \langle \varphi, f_s(\cdot, p) \rangle$ for $\varphi \in (C_b^*(\mathbb{R}_+, E))^n$ and $f \in (C_b(\mathbb{R}_+ \times M; E))^n$.

Lemma 4.40. *If $f \in (C_b(\mathbb{R}_+ \times M; E))^n$ is weakly asymptotically almost periodic with respect to t uniformly with respect to p on compact subsets from M and $\varphi \in (C_b^*(\mathbb{R}_+, E))^n$, $\varphi = (\beta, \beta, \dots, \beta)$ (β is a linear multiplicative functional), then $\widehat{f} \in H^+(f)$. If $\tau_{t_k} f \rightharpoonup g$, then $\tau_{t_k} \widehat{f} \rightharpoonup \widehat{g}$.*

Proof. The set of measures of Dirac $\{\delta_s \mid s \in \mathbb{R}_+\} \subset (C_b^*(\mathbb{R}_+, E))^n$ is dense on the set of all linear continuous multiplicative functionals in weak* topology $(C_b^*(\mathbb{R}_+, E))^n$ [140]. Therefore, there exists $\{t_k\}$ such that $\delta_{t_k} \rightharpoonup \varphi$, that is, $\langle \varphi, f^{(t)}(\cdot, p) \rangle = \lim_{k\to+\infty} \langle \delta_{t_k}, f^{(t)}(\cdot, p) \rangle$. Hence, $\lim_{k\to+\infty} f(t + t_k, p) = \langle \varphi, f^{(t)}(\cdot, p) \rangle$. But that does not mean that $\tau_{t_k} f(t,p) \to \widehat{f}(t,p)$. Extracting, if necessary, a subsequence $\{\tau_{t_{k_m}} f\}$ from $\{\tau_{t_k} f\}$, we obtain $f_{t_{n_k}} \rightharpoonup \widehat{f}$.

Since $(C_b(\mathbb{R}_+, E))^n \ni y \to \widehat{y} \in (C(\mathbb{R}_+, E))^n$ is a linear isometry, we conclude that the second statement of the lemma takes place too. The lemma is proved. □

Lemma 4.41. *If the mapping $p \to f(\cdot, p)$ $(p \in M)$ is continuous, $y \in (C(\mathbb{R}_+, E))^n$, $Q := \overline{y(\mathbb{R}_+)}$ is a compact subset in M and $\varphi \in (C_b^*(\mathbb{R}_+, E))^n$, then $\widehat{f}(s, \widehat{y}(s)) = \langle \varphi, w^{(s)} \rangle$ for $s \in \mathbb{R}_+$, where $w(t) := f(t, y(t))$.*

Proof. Let τ, η, ψ, χ be the mappings defined by the formulas

$$\begin{aligned}
&\tau : C_b^*(\mathbb{R}_+, E) \longrightarrow \left(C_b^*(\mathbb{R}_+, E)\right)^n, && \tau(\varphi) := (\beta(\varphi), \beta(\varphi), \dots, \beta(\varphi)),\\
&\chi : C_b^*(\mathbb{R}_+, E) \longrightarrow E^n, && \chi(\varphi) := \langle \tau(\varphi), w^{(s)} \rangle,\\
&\eta : C_b^*(\mathbb{R}_+, E) \longrightarrow E^n, && \eta(\varphi) := \langle \tau(\varphi), y^{(s)} \rangle,\\
&\psi : C_b^*(\mathbb{R}_+, E) \times E^n \longrightarrow E^n, && \psi(\varphi, p) := \langle \tau(\varphi), f^{(s)}(\cdot, p) \rangle.
\end{aligned} \tag{4.122}$$

Let us consider weak* topology in $C(\mathbb{R}_+, E)$. Then χ, η, ψ are continuous mappings. So, $\beta \to \psi(\beta, \eta(\beta))$ is also continuous. Since on the space of measures of Dirac $\psi(\beta, \eta(\beta)) = \chi(\beta)$, then the same equality takes place on the set of multiplicative functionals too. The lemma is proved. □

4.8. Linear and Semilinear Differential Equations with Weakly Asymptotically Almost Periodic Coefficients

Theorem 4.8.1. *Let $f \in (C_b(\mathbb{R}_+ \times M, E))^n$ be weakly asymptotically almost periodic with respect to t uniformly with respect to $p \in M$. If for every function $g \in \Omega_f := \{g \mid \exists h_k \to +\infty,\ \tau_{h_k} f \rightharpoonup g\}$ the equation*

$$\frac{du}{dt} = g(t, u) \tag{4.123}$$

has at most one solution on $\mathbb{R}$ with the values in the compact set $K \subset M \subset E^n$ and $\psi \in (C_b(\mathbb{R}_+, E))^n$ is a solution of the equation

$$\frac{dx}{dt} = f(t, x) \tag{4.124}$$

such that $\psi(\mathbb{R}_+) \subseteq K$, then ψ is a weakly asymptotically almost periodic function.

Proof. Let $K \subset M$ be a compact set such that $\psi(\mathbb{R}_+) \subseteq K$ and $\{t_k\} \subset \mathbb{R}_+$ ($t_k \to +\infty$). Then there exist a subsequence $\{t_{k_m}\}$ and a function $g \in (C_b(\mathbb{R}_+ \times M, E))^n$ such that $f^{(t_{k_m})} \rightharpoonup g$ and on compact subsets from $\mathbb{R}$ the sequence $\{\psi^{(t_{k_m})}\}$ uniformly converges to the function $y \in (C_b(\mathbb{R}_+, E))^n$ and $y(\mathbb{R}) \subseteq K$. To finish the proof of the theorem it is sufficient to show that $\psi^{(t_{k_m})} \rightharpoonup y$.

Suppose that $\{\psi^{(t_{k_m})}\} \not\rightharpoonup y$. According to Lemma 4.30 there exists a multiplicative functional φ such that $\{\langle \varphi, \psi^{(t_{k_m})} \rangle\}$ does not converge to $\langle \varphi, y \rangle$, where $\varphi \in (C_b^*(\mathbb{R}_+, E))^n$. From the definition of g and y it follows that $\dot{y} = g(t, y)$, and from Lemma 4.34, taking into account the relation $\langle \varphi, \dot{y}^s \rangle = \hat{y}(s)$, we get

$$\hat{y}(t) = \hat{g}(t, \hat{y}(t)) \tag{4.125}$$

for all $t \in \mathbb{R}$. According to Lemma 4.40 $\hat{y} \in H^+(y)$ and, consequently, $\hat{y} \subseteq K$. Similarly, for $t \in \mathbb{R}$

$$\hat{\psi}(t) = \hat{f}(t, \hat{\psi}(t)). \tag{4.126}$$

Since $\{\langle \varphi, \psi^{(t_{k_m})}\rangle\}$ does not converge to $\langle \varphi, y\rangle$, there exist $\varepsilon > 0$ and a subsequence $\{r_m\} \subset \{t_{k_m}\}$ such that

$$|\langle \varphi, \psi^{(r_m)}\rangle - \langle \varphi, y\rangle| \geq \varepsilon \tag{4.127}$$

for all $m \in \mathbb{N}$ and $\{\hat{\psi}^{(r_n)}\}$ uniformly converges to the function $z \in (C_b(\mathbb{R}_+, E))^n$ on every compact subset from $\mathbb{R}$. By Lemma 4.40 $\hat{f}^{(r_m)} \rightharpoonup \hat{g}$ and $\hat{g} \in H^+(f)$. Therefore, taking into account (4.126),

$$\dot{z}(t) = \hat{g}(t, z(t)), \tag{4.128}$$

$(z(\mathbb{R}) \subseteq K)$ for all $t \in \mathbb{R}$. Then from equalities (4.125) and (4.128) it follows that in K the functions z and y are solutions of the same equation, and by the condition of the theorem $z = y$.

On the other hand, from inequality (4.127) we have

$$|z(0) - \hat{y}(0)| = \lim_{m\to+\infty} |\hat{\psi}^{(r_m)}(0) - \langle \varphi, y\rangle| = \lim_{m\to+\infty} |\langle \varphi, \psi^{(r_m)}\rangle - \langle \varphi, y\rangle| \geq \varepsilon. \tag{4.129}$$

The obtained contradiction shows that $\psi^{(t_{k_m})} \rightharpoonup y$. Theorem is proved. □

Lemma 4.42. *Let $I = [a,b] \subset \mathbb{R}$, A, $A_k \in C(I, [E]^n)$ and the following conditions be held:*

(1) $\|A_k(t)\| \leq M$ *for all* $t \in [a,b]$ *and* $k \in \mathbb{N}$, *where* M *is some positive constant;*
(2) $A_k(t) \to A(t)$ *for all* $t \in I$.

Then the following statements hold:

(1) *there exists* $L > 0$ *such that* $\|U(t, A_k)\| \leq L$ *for all* $t \in I$ *and* $k \in \mathbb{N}$, *where* $U(t, A_k)$ *is a Cauchy operator of the equation*

$$\frac{dx}{dt} = A_k(t)x. \tag{4.130}$$

(2) *for every* $t \in I$ $U(t, A_k) \to U(t, A)$ *as* $k \to +\infty$.

Proof. Since $U(t, A_k)$ is a solution of the system

$$\begin{aligned} U'(t, A_k) &= A_k(t)U(t, A_k) \\ U(0, A_k) &= Id_{E^n}, \end{aligned} \tag{4.131}$$

then from [128, inequality (3.1.3)] it follows that

$$||U(t, A_k)|| \leq e^{M[b-a]} := L \quad (k \in \mathbb{N}). \tag{4.132}$$

Let us prove the second statement of the lemma. Assume $V_k(t) = U(t, A) - U(t, A_k)$ and note that $V_k(t)$ satisfies the system

$$\begin{aligned} V_k'(t) &= A(t)V_k(t) + [A(t) - A_k(t)]U(t, A_k) \\ V_k(0) &= 0. \end{aligned} \tag{4.133}$$

Therefore

$$V_k(t) = U(t,A)\int_0^t U^{-1}(\tau,A)[A(\tau) - A_k(\tau)]U(\tau,A_k)d\tau. \tag{4.134}$$

Let $K := \max\{\|U(t,A)\|, \|U^{-1}(t,A)\| : a \le t \le b\}$. From (4.132) and (4.134) it follows the inequality

$$||V_k(t)|| \le K^2L\left|\int_0^t ||A_k(\tau) - A(\tau)||d\tau\right|. \tag{4.135}$$

Passing to the limit in inequality (4.135), taking into consideration the theorem of Lebesgue on the limit passage under the integral sign [142], we will obtain the needed statement. The lemma is proved. □

Lemma 4.43. *If $A \in \mathcal{A}_w(\mathbb{R}_+, [E^n])$ and (3.198) is hyperbolic on $\mathbb{R}_+$, then every equation*

$$\frac{dy}{dt} = B(t)y, \tag{4.136}$$

where $B \in \omega_A = \{B \mid \exists t_k \to +\infty, A^{(t_k)} \rightharpoonup B\}$, is hyperbolic on $\mathbb{R}$.

Proof. Let $B \in \omega_A$. Then there exists $t_k \to +\infty$, $A^{(t_k)} \rightharpoonup B$. Let $P(A)$, $Q(A)$ and N_1, N_2, ν_1, ν_2 be projectors and constants taking part in the definition of the hyperbolicity of (3.198) on $\mathbb{R}_+$. We put

$$P(A^{(t_k)}) = U(t_k,A)P(A)U^{-1}(t_k,A), \tag{4.137}$$

$$Q(A^{(t_k)}) = U(t_k,A)Q(A)U^{-1}(t_k,A). \tag{4.138}$$

From inequalities (3.12) and (3.13), it follows that the operators $P(A^{(t_k)})$ and $Q(A^{(t_k)})$ are uniformly bounded and, consequently, $\{P(A^{(t_k)})\}$ and $\{Q(A^{(t_k)})\}$ can be considered convergent. Assume $P(B) := \lim_{k\to+\infty} P(A^{(t_k)})$ and $Q(B) := \lim_{k\to+\infty} Q(A^{(t_k)})$. Note that

$$P^2(A^{(t_k)}) = P(A^{(t_k)}), \tag{4.139}$$

$$P(A^{(t_k)}) + Q(A^{(t_k)}) = Id_{E^n} \tag{4.140}$$

for all $k \in \mathbb{N}$. Passing to the limit in (4.139) as $k \to +\infty$, we get $P^2(B) = P(B)$. Similarly it is proved that $Q^2(B) = Q(B)$. Finally, from (4.140) it follows that $P(B)+Q(B) = Id_{E^n}$. So, $P(B)$ and $Q(B)$ are a pair of mutually complimentary projectors. Let us show that they can be taken as projectors in the definition of the hyperbolicity on $\mathbb{R}$ of (4.136). In fact, let $t \ge \tau$ and $t, \tau \in \mathbb{R}$. Then for sufficiently large t_k the numbers t and τ belong to the interval $]-t_k, +\infty[$. From the equalities

$$\begin{aligned}
&U(t,A^{(t_k)})P(A^{(t_k)})U^{-1}(\tau,A^{(t_k)})\\
&\quad = U(t,A^{(t_k)})U(t_k,A)P(A)U^{-1}(t_k,A)U^{-1}(\tau,A^{(t_k)})\\
&\quad = U(t+t_k,A)P(A)U^{-1}(\tau+t_k,A)
\end{aligned} \tag{4.141}$$

and inequality (3.12), taking into account the above said and Lemma 4.42, we get the inequality $\|U(t,B)P(B)U^{-1}(\tau,B)\| \le N_1 e^{-\nu_1(t-\tau)}$.

Similarly it is proved that $\|U(t,B)Q(B)U^{-1}(\tau,B)\| \le N_2 e^{\nu_2(t-\tau)}$ for $t \le \tau$ and $t,\tau \in \mathbb{R}$. The lemma is proved. □

Corollary 4.44. *Let* $A \in \mathcal{A}_w(\mathbb{R}_+,[E^n])$ *and* (4.136) *be hyperbolic on* $\mathbb{R}_+$. *Then for every* $B \in \omega_A$ (4.123) *has no nonzero bounded on* $\mathbb{R}$ *solutions.*

Theorem 4.8.2. *Let* (3.198) *be hyperbolic on* $\mathbb{R}_+$ *and* $A(t) \in \mathcal{A}_w(\mathbb{R}_+,[E^n])$. *If* $f \in \mathcal{A}_w(\mathbb{R}_+,[E^n])$, *then* (3.200) *has at least one weakly asymptotically almost periodic solution. This solution is defined by equality* (3.202).

Proof. Note that every weakly asymptotically almost periodic function $f \in \mathcal{A}_w$ is bounded on $\mathbb{R}_+$, that is why from [120] it follows that by equality (3.202) there is given a bounded on $\mathbb{R}_+$ solution of (3.200). According to Corollary 4.44 for every $B \in \omega_A$ and $g \in \omega_f$ (3.201) has no more than one bounded on $\mathbb{R}$ solution. Then by Theorem 4.8.1 this solution is weakly asymptotically almost periodic. □

Theorem 4.8.3. *Let* $A \in \mathcal{A}_w(\mathbb{R}_+,[E^n])$ *and* (3.198) *be hyperbolic on* $\mathbb{R}_+$. *If* $F \in C_b(\mathbb{R}_+ \times E^n, E^n)$ *is weakly asymptotically almost periodic with respect to t uniformly with respect to x on compact subsets from* E^n *and satisfies the condition of Lipschitz with respect x uniformly with respect to* $t \in \mathbb{R}_+$ *with a small enough constant of Lipschitz, then* (3.282) *has at least one weakly asymptotically almost periodic solution.*

Proof. Let us consider the mapping

$$\Phi : \mathcal{A}_w(\mathbb{R}_+,E^n) \longrightarrow \mathcal{A}_w(\mathbb{R}_+,E^n) \tag{4.142}$$

defined by the equality

$$(\Phi y)(t) = \int_0^{+\infty} G_A(t,\tau)F(\tau,y(\tau))\,d\tau. \tag{4.143}$$

According to Lemma 4.34 and Theorem 4.8.2, the equality (4.143) well defines the operator $\Phi : \mathcal{A}_w(\mathbb{R}_+,E^n) \to \mathcal{A}_k(\mathbb{R}_+,E^n)$. From the estimation $\|G_A(t,\tau)\| \le Ne^{-\nu|t-\tau|}$ and the condition of Lipschitz for F it follows that Φ is a contractive mapping, if $(2N/\nu)L < 1$, where L is the constant of Lipschitz of the function F. Then the single fixed point of the mapping Φ will be a weakly asymptotically almost periodic solution of (3.282). The theorem is proved. □

Corollary 4.45. *Let* $A \in \mathcal{A}_{wk}(\mathbb{R}_+,[E^n])$, (3.198) *be hyperbolic on* $\mathbb{R}_+$ *and* $F \in C_b(\mathbb{R}_+ \times E^n, E^n)$ *be weakly asymptotically almost periodic with respect to t uniformly with respect to x on compact subsets from* E^n *and satisfy the condition of Lipschitz with respect x uniformly with respect to* $t \in \mathbb{R}_+$. *Then there exists a number* $\varepsilon_0 > 0$ *such that for every* $|\varepsilon| \le \varepsilon_0$ *equation*

$$\frac{dx}{dt} = A(t)x + \varepsilon F(t,x) \tag{4.144}$$

has at least one weakly asymptotically almost periodic solution φ_ε, *and* $\|\varphi_\varepsilon\| \to 0$ *as* $\varepsilon \to 0$.

Remark 4.46. Every asymptotically almost periodic in the sense of Fréchet function is weakly asymptotically almost periodic. Further, if $p \in C_b(\mathbb{R}, E^n)$ is weakly almost periodic [139] (i.e., the set of all shifts $\{p_\tau \mid \tau \in \mathbb{R}\}$ of the function p is relatively compact in the weak* topology $C_b(\mathbb{R}, E^n)$) and $\omega \in C_0(\mathbb{R}, E^n)$, then the function $\varphi = p + \omega$ is weakly asymptotically almost periodic. Note that there exist weakly asymptotically almost periodic functions that are not asymptotically almost periodic in the sense of Fréchet. To confirm the above said it is enough to consider the function $\varphi(t) := \sin(t+\ln(1+|t|))+e^{-t}$. The function $p(t) := \sin(t+\ln(1+|t|))$ is weakly almost periodic but not almost periodic in the sense of Bohr (see, i.e., [139]).

5 Asymptotically Almost Periodic Solutions of Functionally Differential, Integral, and Evolutionary Equations

5.1. Functional Differential Equations (FDEs) and Dynamical Systems

Let $r > 0$, $C([a,b],E^n)$ be a Banach space of all continuous functions $\varphi : [a,b] \to E^n$ with the norm sup. If $[a,b] := [-r,0]$, then assume $\mathcal{C} := C([-r,0],E^n)$. Let $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $u \in C([\alpha - r, \alpha + \beta], E^n)$. For every $t \in [\alpha, \alpha + \beta]$ define $u_t \in \mathcal{C}$ by the relation $u_t(\theta) := u(t + \theta)$, $-r \leq \theta \leq 0$.

Example 5.1 (Autonomous functionally differential equations (autonomous FDEs)). Consider a differential equation

$$\frac{dx(t)}{dt} = f(x_t), \tag{5.1}$$

where $f \in C(\mathcal{C}, E^n)$. Concerning (5.1) we will suppose that the conditions of existence, uniqueness and nonlocally continuability of solutions on $\mathbb{R}_+$ are fulfilled. Let $\varphi \in \mathcal{C}$ and x be the solution of (5.1) satisfying the initial condition

$$x(s) = \varphi(s) \quad (s \in [-r,0]). \tag{5.2}$$

Define a mapping $\pi : \mathbb{R}_+ \times \mathcal{C} \to \mathcal{C}$ by the rule $\pi(t,\varphi) = x_t$, where x is the solution of the Cauchy problem (5.1)–(5.2). From the general properties of FDEs [143, 144] it follows that π is continuous $\pi(0,\varphi) = \varphi$ $(\varphi \in C)$ and $\pi(t_2, \pi(t_1,\varphi)) = \pi(t_1 + t_2, \varphi)$ for all $\varphi \in \mathcal{C}$ and $t_1, t_2 \in \mathbb{R}_+$ and, consequently, $(\mathcal{C}, \mathbb{R}_+, \pi)$ is a semigroup dynamical system on $\mathcal{C}$.

Example 5.2 (Nonautonomous FDEs with uniqueness). Denote by $C(\mathbb{R} \times \mathcal{C}, E^n)$ the set of all continuous functions $f : \mathbb{R} \times \mathcal{C} \to E^n$ with the compact-open topology and by $(C(\mathbb{R} \times \mathcal{C}, E^n), \mathbb{R}, \sigma)$ the dynamical system of shifts on $C(\mathbb{R} \times \mathcal{C}, E^n)$ (see Example 1.47). Let us consider a differential equation

$$\frac{dx(t)}{dt} = f(t, x_t), \tag{5.3}$$

where $f \in C(\mathbb{R} \times \mathcal{C}, E^n)$.

Definition 5.3. The function $f \in C(\mathbb{R} \times \mathcal{C}, E^n)$ is called regular, if for every $g \in H(f) := \overline{\{f^{(\tau)} : \tau \in \mathbb{R}\}}$, where $f^{(\tau)} := \sigma(\tau, f)$, for the equation

$$\frac{dy(t)}{dt} = g(t, y_t), \tag{5.4}$$

there are held the conditions of the existence, uniqueness and nonlocally continuability of solutions on $\mathbb{R}_+$.

Let $f \in C(\mathbb{R} \times \mathcal{C}, E^n)$ be regular. Put $Y := H(f)$ and by $(Y, \mathbb{R}, \sigma)$ denote the dynamical system of shifts on Y induced by the dynamical system $(C(\mathbb{R} \times \mathcal{C}, E^n), \mathbb{R}, \sigma)$. Define a mapping $\pi : \mathbb{R}_+ \times X \to X$, where $X := \mathcal{C} \times Y$, by the equality $\pi(\tau, (\psi, g)) := (y_\tau, g^{(\tau)})$, where y is the solution of (5.4) satisfying the initial condition

$$y(s) = \psi(s) \quad (s \in [-r, 0]). \tag{5.5}$$

From the theorem on continuous dependence of solutions on the initial data and the right-hand side (see, i.e., [143, Chapter 2]) it follows that the mapping π is continuous. Further, assume $\phi(\tau, \psi, g) = y_\tau$, where $y : [-r, +\infty[\to E^n$ is the solution of (5.4) satisfying the condition (5.5) and $y_\tau \in \mathcal{C}$ is defined by the equality $y_\tau(s) := y(s + \tau)$ $(s \in [-r, 0])$. It is easy to verify that the equality $\phi(t, \phi(\tau, \psi, g), g_\tau) = \phi(t + \tau, \psi, g)$ takes place for all $t, \tau \in \mathbb{R}_+$, $\psi \in \mathcal{C}$ and $g \in H(f)$. Therefore $\pi(\tau, \pi(t, x)) = \pi(t + \tau, x)$ for all $t, \tau \in \mathbb{R}_+$ and $x \in X = \mathcal{C} \times H(f)$. At last, note that $\pi(0, x) = x$ for all $x \in X = \mathcal{C} \times H(f)$ and, consequently, $(X, \mathbb{R}_+, \pi)$ is a semigroup dynamical system on $X := \mathcal{C} \times H(f)$. Assume $h := pr_2 : X \to Y$. It is easy to verify that h is a homomorphism of $(X, \mathbb{R}_+, \pi)$ onto $(Y, \mathbb{R}, \sigma)$ and, consequently, the triplet $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a nonautonomous dynamical system generated by (5.3) with a regular right-hand side f.

Example 5.4 (Nonautonomous FDEs without uniqueness). Let $\mathbb{R}_r := [-r, +\infty[$ and $C(\mathbb{R}_r, E^n)$ be the space of all continuous functions $f : \mathbb{R}_r \to E^n$ with the topology of uniform convergence on compacts and $(C(\mathbb{R}_r, E^n), \mathbb{R}_+, \sigma)$ be a dynamical system of shifts on $C(\mathbb{R}_r, E^n)$ (see Example 1.46). Assume $Y := C(\mathbb{R} \times \mathcal{C}, E^n)$ and by $(Y, \mathbb{R}, \sigma)$ denote a dynamical system of shifts on $C(\mathbb{R} \times \mathcal{C}, E^n)$. Further, let $X := \{(\varphi, f) : \varphi \in C(\mathbb{R}_r, E^n),\ f \in C(\mathbb{R} \times \mathcal{C}, E^n)$, and φ be a solution of (5.3)$\}$. Obviously, X is positively invariant (with respect to shifts) set of the product dynamical system $(C(\mathbb{R}_r, E^n), \mathbb{R}_+, \sigma) \times (C(\mathbb{R} \times \mathcal{C}, E^n), \mathbb{R}, \sigma)$. Besides, the results of works [145, 143] imply that X is closed in $C(\mathbb{R}_r, E^n) \times C(\mathbb{R} \times \mathcal{C}, E^n)$ and, consequently, on X there is induced a semigroup dynamical system $(X, \mathbb{R}_+, \pi)$. It is easy to see that the mapping $h := pr_2 : X \to Y$ is a homomorphism of the dynamical system $(X, \mathbb{R}_+, \pi)$ onto $(Y, \mathbb{R}, \sigma)$ and, consequently, $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a nonautonomous dynamical system generated by (5.3) the right-hand side of which is not regular.

In the previous examples there was realized the concept of FDEs with finite delay. In the next example we will consider an FDE with nonlimited delay. But previously we will introduce some functional spaces.

5.1.1. The Space of Hale

Let $\mathbb{B}$ be a vector space of functions $\phi : \mathbb{R}_- \to E^n$ ($\mathbb{R}_- =]-\infty, 0]$) with the seminorm $|\cdot|_{\mathbb{B}}$.

For $\beta \geq 0$ and $\phi \in \mathbb{B}$ by ϕ^β denote the restriction ϕ on $]-\infty, -\beta]$ and $\mathbb{B}^\beta := \{\phi^\beta | \phi \in \mathbb{B}\}$. On $\mathbb{B}^\beta$ define the seminorm $|\cdot|_\beta$ by the equality

$$|\eta|_\beta := \inf\{|\psi|_{\mathbb{B}} : \psi \in \mathbb{B}\ \psi^\beta = \eta\}. \tag{5.6}$$

If $x :]-\infty, a[\to E^n$ $(a > 0)$, then for every $t \in [0, a[$ we can define a function x_t by the relation $x_t(s) := x(t+s)$ $(s \in \mathbb{R}_-)$. For numbers a and τ $(a > \tau)$ by A_τ^a we denote the class of functions $x :]-\infty, a[\to E^n$ such that x is continuous on $[\tau, a[$ and $x_\tau \in \mathbb{B}$.

Definition 5.5. B is called a space of Hale (see, e.g., [146]), if the following conditions are fulfilled:

(1) if $x \in A_\tau^a$, then $x_\tau \in \mathcal{B}$ for all $t \in [\tau, a[$ and x_t is continuous with respect to t;
(2) for every $\phi \in \mathbb{B}$ and $\beta \geq 0$ if $|\phi|_{\mathbb{B}} = 0$, then $|\tau^\beta \phi|_\beta = 0$, where τ^β is the linear operator acting from $\mathbb{B}$ to B^β and defined by the equality $\tau^\beta\phi(\theta) := \phi(\theta + \beta)$ $(\theta \in]-\infty, -\beta])$;
(3) if the sequence $\{\phi_k\} \subseteq \mathbb{B}$ is uniformly bounded on $\mathbb{R}_-$ with respect to the seminorm $|\cdot|_{\mathbb{B}}$ and converges to ϕ uniformly on compact subsets of $\mathbb{R}_-$, then $\phi \in \mathbb{B}$ and $|\phi_k - \phi|_{\mathbb{B}} \to 0$, when $k \to +\infty$;
(4) there exists a number $K > 0$ such that for all $\phi \in \mathbb{B}$ and $\beta \geq 0$

$$|\phi|_{\mathbb{B}} \leq K\Big(\sup_{-r\leq\theta\leq 0} |\phi(\theta)| + |\phi^\beta|_\beta\Big); \tag{5.7}$$

(5) if $\phi \in \mathbb{B}$, then $|\tau^\beta\phi|_\beta \to 0$ as $\beta \to +\infty$;
(6) $|\phi(0)| \leq M_1|\phi|_{\mathbb{B}}$ for some $M_1 > 0$.

5.1.2. Examples of Hales's Spaces

(a) $C_{bu}(\mathbb{R}_-, E^n) := \{\phi \mid \phi : \mathbb{R}_- \to E^n$, ϕ is uniformly continuous and bounded$\}$ with the norm sup.
(b) $C_\nu := \{\phi \mid \phi : \mathbb{R}_- \to E^n$, ϕ is continuous, $\phi(\theta)e^{\nu\theta} \to 0$ as $\theta \to -\infty\}$ with the norm $|\phi|_{C_\nu} := \sup\{|\phi(\theta)|e^{\nu\theta} : \theta \in]-\infty, 0]\}$.
(c) Let $r \geq 0$, $p \geq 1$ and $g(\theta)$ be a nondecreasing function, positive, and defined on $\mathbb{R}_-$, satisfying the condition $\int_{-\infty}^0 g(\theta)d\theta < +\infty$. $\mathbb{B}$ consists of measurable in the sense of Lebesgue mappings $\phi : \mathbb{R}_r \to E^n$ continuous on $[-r, 0]$ with the norm

$$|\phi|_{\mathbb{B}} := \Big\{\sup_{-r\leq\theta\leq 0} |\phi(\theta)|^p + \int_{-\infty}^0 |\phi(\theta)|^p g(\theta)d\theta\Big\}^{1/p}. \tag{5.8}$$

Example 5.6 (FDEs with unlimited delay). Let $\mathbb{B}$ be a space of Hale and $W \subseteq \mathbb{B}$. Consider differential (5.3) where $f \in C(\mathbb{R} \times W, E^n)$. As well as in the case of FDEs from Examples 5.2 and 5.4, under some standard assumptions, by (5.3) we can construct two nonautonomous dynamical systems: the first one when the right-hand side f is regular and the second one when it is not.

5.2. Asymptotically Almost Periodic Solutions of FDEs

Applying the results of Chapter 2 to nonautonomous dynamical systems constructed in Examples 5.2–5.6 (as it was done in Chapter 3 for ordinary differential equations), we can obtain series of tests of the existence of asymptotically almost periodic solutions for FDEs with finite and infinite delay.

Definition 5.7. A solution $\phi \in C(\mathbb{T}, E^n)$ ($\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$) of (5.1) one will call compact on $\mathbb{T}$, if the set $\{\sigma(\tau, \phi) := \phi^\tau \mid \tau \in \mathbb{T}\}$ (where ϕ^τ is the shift of the function ϕ on τ) is relatively compact in $C(\mathbb{T}, E^n)$.

As we know [143], this will take place if and only if the function ϕ is bounded and uniformly continuous on $\mathbb{T}$.

Let $\phi \in C(\mathbb{T}, E^n)$ and the set $\{\sigma(\tau, \phi) \mid \tau \in \mathbb{T}\}$ be relatively compact in $C(\mathbb{T}, E^n)$. Assume $Q_\phi^{\mathbb{T}} := \overline{\{\widetilde{\phi}_\tau \mid \tau \in \mathbb{T}\}}$, where $\widetilde{\phi}_\tau := \phi_\tau|_{[-r,0]} \in C([-r, 0], E^n)$ and by bar it is denoted the closure in $\mathcal{C}$. Then $Q_\phi^{\mathbb{T}}$ is a compact subset in $\mathcal{C}$. Put $Q_\phi^+ := Q_\phi^{\mathbb{R}_+}$ and $Q_\phi := Q_\phi^{\mathbb{R}}$.

Theorem 5.2.1. *Let $\phi \in C(\mathbb{R}_r, E^n)$ be a compact on $\mathbb{R}_+$ solution of* (5.3) *and f be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) with respect to the variable $t \in \mathbb{R}$ uniformly with respect to $\varphi \in Q_\phi^+$. If for every $g \in \omega_f$* (5.4) *admits at most one solution from ω_ϕ, then ϕ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).*

Let $\phi \in C(\mathbb{R}, E^n)$ and $M \subset C(\mathbb{R}, E^n)$.

Definition 5.8. One will say that the function ϕ is separated in M (see Section 3.6), if M consists only from the function ϕ or if there exists a number $r > 0$ such that for every function $\mu \in M$ ($\mu \neq \varphi$) there is fulfilled the inequality

$$\max_{-r \le \theta \le 0} \left| \phi(t+\theta) - \mu(t+\theta) \right| \ge r. \tag{5.9}$$

Theorem 5.2.2. *Let $\phi \in C(\mathbb{R}_r, E^n)$ be a compact on $\mathbb{R}_+$ solution of* (5.3) *and f be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) with respect to $t \in \mathbb{R}$ uniformly with respect to $\varphi \in Q_\phi^+$. If for every $g \in \omega_f$ all solutions from ω_ϕ of* (5.4) *are separated in ω_ϕ, then ϕ is asymptotically stationary (resp., asymptotically $k_0\tau$-periodic for some natural k_0, asymptotically almost periodic, asymptotically recurrent).*

Theorem 5.2.3. *Let $\phi \in C(\mathbb{R}_r, E^n)$ be compact on $\mathbb{R}_+$ solution of* (5.1), *f be asymptotically τ-periodic with respect to $t \in \mathbb{R}$ uniformly with respect to $\varphi \in Q_\phi^+$, and $\overline{g}(t, \varphi) :=$* $\lim_{k \to +\infty} f(t + k\tau, \varphi)$ *(uniformly with respect to $t \in [0, \tau]$ and $\varphi \in Q_\varphi^+$). If the equation*

$$\frac{dy}{dt} = \overline{g}(t, y_t) \tag{5.10}$$

admits at most one solution from ω_ϕ, the solution ϕ is asymptotically τ-periodic.

Denote by $\mathfrak{D} := \mathfrak{D}(\mathcal{C}, E^n)$ the Banach space of all linear continuous operators $\mathcal{C} \to E^n$ with the operator norm. Let us consider a linear equation

$$\frac{dx}{dt} = A(t, x_t), \tag{5.11}$$

where $A : \mathbb{R} \times \mathcal{C} \to E^n$ is continuous and linear with respect to the second variable, that is, $A \in C(\mathbb{R}, \mathfrak{D})$. Along with (5.11) let us consider the corresponding nonhomogeneous equation

$$\frac{dx}{dt} = A(t, x_t) + f(t), \tag{5.12}$$

where $f \in C(\mathbb{R}, E^n)$.

For $A \in C(\mathbb{R}, \mathfrak{D})$ denote by ω_A its ω-limit set in the dynamical system of shifts $(C(\mathbb{R}, \mathfrak{D}), \mathbb{R}, \sigma)$.

Theorem 5.2.4. *Let ϕ be compact on $\mathbb{R}_+$ solution of* (5.12), *$A \in C(\mathbb{R}, \mathfrak{D})$, and $f \in C(\mathbb{R}, E^n)$ be jointly asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent). If every equation of the family*

$$\frac{dy}{dt} = B(t, y_t), \tag{5.13}$$

where $B \in \omega_A$, has no nontrivial compact on $\mathbb{R}$ solutions, then ϕ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).

Related to Theorem 5.2.4 naturally arises the following question: under the conditions of Theorem 5.2.3, will (5.12) admit at least one compact on $\mathbb{R}_+$ solution? The answer to this question follows from the results given in the next chapter.

5.3. Linear FDEs

Using some ideas and methods developed for the study of dissipative dynamical systems, we can obtain series of conditions equivalent to the asymptotical stability of linear nonautonomous dynamical system with infinite-dimensional phase space. As applications we will get the according statements for linear FDEs.

Let (X, h, Y) be a vectorial fiber bundle with the fiber E (E is a Banach space) and $\|\cdot\| : X \to \mathbb{R}_+$ is the norm on X compatible with the metric X, that is, $\|\cdot\|$ is continuous and $\|x\| := \rho(x, \theta_y)$, where $x \in X_y$, θ_y is the zero element of X_y and ρ is a metric on X.

Definition 5.9. The system $(X, \mathbb{S}_+, \pi)$ one will call locally compact (completely continuous), if for every $x \in X$ there exist $\delta_x > 0$ and $l_x > 0$ such that the set $\pi^t B(x, \delta_x)$ $(t \geq l_x)$ is relatively compact.

Definition 5.10. Recall [98, 109] that the nonautonomous system $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}_+, \sigma), h \rangle$ is called linear, if (X, h, Y) is a vectorial fibering and for every $y \in Y$ and $t \in \mathbb{S}_+$ the mapping $\pi^t : X_y \to X_{\sigma(y,t)}$ is linear.

Theorem 5.3.1. *If the linear nonautonomous system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S}_+,\sigma),h\rangle$ is locally compact (i.e., $(X,\mathbb{S}_+,\pi)$ is locally compact) and Y is compact, then the next conditions are equivalent:*

(1) $\lim_{t\to+\infty}\|xt\| = 0$ *for all* $x\in X$*;*
(2) *all motions* $(X,\mathbb{S}_+,\pi)$ *are relatively compact and in* $(X,\mathbb{S}_+,\pi)$ *there is no nontrivial compact continuable onto* $\mathbb{S}$ *motions;*
(3) *there exist positive numbers* N *and* ν *such that* $\|xt\| \le Ne^{-\nu t}\|x\|$ *for all* $x\in X$*,* $t\in\mathbb{S}_+$.

Proof. The equality $\lim_{t\to+\infty}\|xt\| = 0$ imply that Σ_x^+ is relatively compact and $\omega_x \subseteq \theta = \{\theta_y \mid y\in J_Y$, where θ_y is the zero element of X_y and J_Y is the Levinson center of the dynamical system $(Y,\mathbb{S}_+,\sigma)\}$. So, the dynamical system $(X,\mathbb{S}_+,\pi)$ is point dissipative and according to [112] is compactly dissipative. Denote by J_X the Levinson center of the dynamical system $(X,\mathbb{S}_+,\pi)$ and we will show that $J_X = \theta$. Obviously, θ is compact and invariant set and, consequently, $\theta\subseteq J_X$. From the last inclusion it follows that $h(J_X) = J_Y$. If we suppose that $J_X \ne \theta$, then $J_X\backslash\theta \ne \varnothing$ and hence there is $x_0\in J_X\backslash\theta$. Since in J_X all motions are continuable onto $\mathbb{S}$ [109, 113], there exists a continuous mapping $\varphi:\mathbb{S}\to J_X$ such that $\varphi(0) = x_0$ and $\pi^t\varphi(s) = \varphi(t+s)$ for all $s\in\mathbb{S}$ and $t\in\mathbb{S}_+$. On the other hand, in virtue of the linearity of the system $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S}_+,\sigma),h\rangle$ along with the point x_0 all points λx_0 also belong to the set J_X ($\lambda\in\mathbb{R}$), as J_X is the maximal compact invariant set in X. But $\lambda x_0\in J_X$ for all $\lambda\in\mathbb{R}$ if and only if $x_0\in\theta$. The obtained contradiction shows that $J_X = \theta$. So, in $(X,\mathbb{S}_+,\pi)$ there is no nontrivial compact continuable onto $\mathbb{S}$ motions (since they all are in J_X). So, we showed that from (1) it follows that (2).

Let us prove that (2) implies (3). Let condition (2) be fulfilled. Then the system $(X,\mathbb{S}_+,\pi)$ is locally dissipative. By the compactness of Y and local dissipativity of $(X,\mathbb{S}_+,\pi)$ there is $\delta > 0$ such that

$$\lim_{t\to+\infty}\sup\{\|xt\| : \|x\| < \delta\} = 0. \tag{5.14}$$

From (5.14) by standard reasoning (see, i.e., [109, 122, 147]) we can show that there are $N,\nu > 0$ such that $\|xt\| \le Ne^{-\nu t}\|x\|$ for all $x\in X$ and $t\in S_+$. At last, it is obvious that (1) follows from (3). The theorem is proved. □

Linear nonautonomous FDEs [143, 144] present an important classes of linear nonautonomous systems with infinite-dimensional phase space satisfying the condition of local completely continuity.

Along with (5.11) we consider the family of (5.13), where $B\in H^+(A) := \overline{\{A^{(\tau)} : \tau\in\mathbb{R}_+\}}$, by bar there is denoted the closure in $C(\mathbb{R},\mathfrak{D})$ ($C(\mathbb{R},\mathfrak{D})$ is endowed with the topology of uniform convergence on compact subsets from $\mathbb{R}$) and $A^{(\tau)}(t) = A(t+\tau)$.

Let $\phi(t,\varphi,B)$ be the solution of (5.13) passing through the point $\varphi\in\mathfrak{C}$ as $t = 0$, defined for all $t\in\mathbb{R}_+$. Assume $Y = H^+(A)$ and by $(Y,\mathbb{R}_+,\sigma)$ denote the semigroup dynamical system of shifts on $H^+(A)$. Let $X := \mathfrak{C}\times Y$, $(X,\mathbb{R}_+,\pi)$ be the semigroup dynamical system on X defined by the following rule: $\pi(\tau,(\varphi,B)) := (\phi(\tau,\varphi,B),B^\tau)$ and $h := pr_2 : X\to Y$. Then the nonautonomous system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\sigma),h\rangle$ is linear. Let us notice one important property of the constructed nonautonomous dynamical system. There takes place the following lemma.

Lemma 5.11. *Let $H^+(A)$ be compact in $C(\mathbb{R}, \mathfrak{D})$. Then for every point $x \in X = \mathcal{C} \times H^+(A)$ there exists a neighborhood U_x of the point x and a number $l_x > 0$ such that $\pi^t U_x$ is relatively compact for all $t \geq l_x$, that is, the dynamical system $(X, \mathbb{R}_+, \pi)$ is locally completely continuous.*

Proof. The formulated statement follows from [143, Lemmas 2.2.3 and 3.6.1] and from the compactness of $H^+(A)$. □

Applying Theorem 5.3.1 to the constructed linear nonautonomous dynamical system and taking into consideration Lemma 5.11, we will get the next statement.

Theorem 5.3.2. *Let $H^+(A)$ be compact. Then the following statements are equivalent:*

(1) *the zero solution of (5.11) is uniformly exponentially stable, that is, there exist positive numbers N and $\nu > 0$ such that $\|\phi(t, \varphi, B)\| \leq Ne^{-\nu t}\|\varphi\|$ for all $\varphi \in \mathcal{C}$, $B \in H^+(A)$ and $t \in \mathbb{R}_+$;*

(2) *for any $B \in H^+(A)$ the zero solution of (5.13) is asymptotically stable;*

(3) *for any $B \in H^+(A)$ all solutions of (5.13) are compact (bounded) on $\mathbb{R}_+$ and for any $B \in \omega_A$ (5.13) has no nonzero compact (bounded) on $\mathbb{R}$ solutions.*

Remark 5.12. (1) The nonzero solution of (5.11) is uniformly exponentially stable if and only if there exist positive numbers N and ν such that $\|\phi(t, \varphi, A^{(\tau)})\| \leq Ne^{-\nu t}\|\varphi\|$ for all $\varphi \in \mathcal{C}$ and $t, \tau \in \mathbb{R}_+$.

(2) Let exist positive numbers N and ν such that $\|\phi(t, \varphi, A^{(\tau)})\| \leq Ne^{-\nu t}\|\varphi\|$ for all $\varphi \in \mathcal{C}$ and $t, \tau \in \mathbb{R}_+$. Then $\|\phi(t, \varphi, A)\| \leq Ne^{-\nu(t-\tau)}\|\phi(\tau, \varphi, A)\|$ for all $\varphi \in \mathcal{C}$ and $t \geq \tau$ $(t, \tau \in \mathbb{R}_+)$.

(3) Let $(X, \mathbb{S}_+, \pi)$ be a dynamical system, X be compact and $X := H^+(x_0) = \overline{\{x_0 t \mid t \in S_+\}}$, where $x_0 \in X$. Then $(X, \mathbb{S}_+, \pi)$ is compactly dissipative and $J_X = \omega_{x_0}$, where J_X is the Levinson center of $(X, \mathbb{S}_+, \pi)$.

5.4. Semilinear FDEs

Denote by $U_t(\cdot, s)$ the Cauchy operator [143, 144] (fundamental matrix) of (5.11) and by $\phi(t, \varphi, A, f)$ the solution of (5.12) passing through the point $\varphi \in C$ as $t = 0$. Then according to the formula of variation of constants (see, e.g., [143, page 177]):

$$\phi(t, \varphi, A, f) = \phi(t, \varphi, A) + \int_0^t U_t(\cdot, s) f(s) ds. \tag{5.15}$$

Lemma 5.13. *Let exist positive numbers N and ν such that*

$$||\phi(t, \varphi, A)|| \leq Ne^{-\nu(t-\tau)}||\phi(\tau, \varphi, A)|| \tag{5.16}$$

for all $t \geq \tau \geq 0$ and $\varphi \in \mathcal{C}$. If $f \in C_b(\mathbb{R}_+, E^n)$, then

(1) *all solutions of (5.12) are bounded on $\mathbb{R}_+$;*

(2) *the solution $\phi(t, 0, A, f) = \int_0^t U_t(0, s) f(s) ds$ of (5.12) can be estimated as:*

$$\|\phi\|_{C_b(\mathbb{R}_+, C)} \leq e^{\nu r} \frac{N}{\nu} \|f\|_{C_b(\mathbb{R}_+, E^n)}. \tag{5.17}$$

Proof. Under the conditions of the lemma we have

$$
\begin{aligned}
||\phi(t,0,A,f)|| &= \left|\left| \int_0^t U_t(\cdot,s)f(s)ds \right|\right| \le \int_0^t ||U_t(\cdot,s)|| \,|\, f(s) \,|\, ds \le \int_0^t Ne^{-\nu(t-s)}e^{\nu r} \,|\, f(s) \,|\, ds \\
&\le Ne^{\nu r} \sup_{t\ge 0} |\, f(t) \,| \frac{e^{-\nu(t-s)}}{\nu} |_0^t = Ne^{\nu r} \|f\|_{C_b(\mathbb{R}_+,E^n)} \frac{1-e^{-\nu t}}{\nu} \\
&\le \frac{N}{\nu} e^{\nu r} \|f\|_{C_b(\mathbb{R}_+,E^n)}.
\end{aligned}
\tag{5.18}
$$

So, we established (5.17).

The first statement of the lemma follows from the formula (5.15) and inequalities (5.16) and (5.17). □

Remark 5.14. If $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$, the operator $A \in C(\mathbb{R}, \mathfrak{D})$ and $f \in C(\mathbb{R}, E^n)$ are bounded on $\mathbb{T}$ and $\varphi : \mathbb{T} \to \mathcal{C}$ is a bounded on $\mathbb{T}$ solution of (5.12), then φ is compact on $\mathbb{T}$.

The formulated statement follows from the theorem of Artzela-Ascoli.

Corollary 5.15. *Under the conditions of Lemma 5.15, if the operator $A \in C(\mathbb{R}, \mathfrak{D})$ is bounded on $\mathbb{R}_+$, then all solutions of* (5.12) *are compact on $\mathbb{R}_+$.*

Theorem 5.4.1. *Let $A \in C(\mathbb{R}, \mathfrak{D})$ and $f \in C(\mathbb{R}, E^n)$ be jointly asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) and the zero solution of* (5.11) *is uniformly exponentially stable, that is, there exist positive numbers N and ν such that*

$$||\phi(t,\varphi,A_s)|| \le Ne^{-\nu t}\|\varphi\| \tag{5.19}$$

for all $t,s \in \mathbb{R}_+$ and $\varphi \in C$. Then for any $\varphi \in \mathcal{C}$ the solution $\phi(t,\varphi,A,f)$ of (5.12) *is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).*

Proof. From inequality (5.19), according to Corollary 5.12, it follows (5.16) and by Lemma 5.13 all solutions of (5.12) are bounded on $\mathbb{R}_+$. Moreover, from Remark 5.14 and Corollary 5.15 it follows that all solutions of (5.12) are compact on $\mathbb{R}_+$. According to Lemma 5.3.2 for any $B \in \omega_A$ (5.13) has no nonzero compact on $\mathbb{R}$ solutions. Now to complete the proof of the theorem it is enough to refer to Theorem 5.2.4. □

Theorem 5.4.2. *Let $A \in C(\mathbb{R}, \mathfrak{D})$ and $f \in C(\mathbb{R}, E^n)$ be asymptotically almost periodic functions, and function $F \in C(\mathbb{R} \times C, E^n)$ be asymptotically almost periodic w.r.t $t \in \mathbb{R}$ uniformly with respect to φ on compacts from C, and let it satisfy the condition of Lipschitz with respect to $\varphi \in \mathcal{C}$ with the constant of Lipschitz $L < (\nu/N)e^{-\nu r}$ (constants N and ν is from* (5.19)*). If the zero solution of* (5.11) *is uniformly exponentially stable, then the equation*

$$\frac{dx(t)}{dt} = A(t)x_t + f(t) + F(t,x_t). \tag{5.20}$$

has at least one asymptotically almost periodic solution.

Proof. Denote by $AP(\mathbb{R}_+, \mathfrak{C})$ the Banach space of all asymptotically almost periodic functions $\varphi : \mathbb{R}_+ \to \mathfrak{C}$ with the norm sup. Define an operator $\Phi : AP(\mathbb{R}_+, \mathfrak{C}) \to AP(\mathbb{R}_+, \mathfrak{C})$ by the following rule: $\Phi(\psi) := \varphi$, where $\psi \in AP(\mathbb{R}_+, \mathfrak{C})$ and

$$(\Phi\psi)(t) = \int_0^t U_t(\cdot, s)[F(s, \psi_s) + f(s)]ds, \tag{5.21}$$

that is, φ is a unique asymptotically almost periodic solution of the equation

$$\frac{dx}{dt} = A(t)x_t + f(t) + F(t, \psi_t) \tag{5.22}$$

satisfying the initial condition $\varphi(0) = 0$.

Let us show that the mapping Φ is contracting. In fact, let $\psi_1, \psi_2 \in AP(\mathbb{R}_+, E^n)$ and $\varphi := \varphi_1 - \varphi_2 = \Phi(\psi_1) - \Phi(\psi_2)$. Then

$$\varphi'(t) = A(t)\varphi_t + F(t, \psi_{1t}) - F(t, \psi_{2t}) \tag{5.23}$$

and $\varphi(0) = 0$. According to Lemma 5.13

$$\begin{aligned} \|\varphi\|_{AP(\mathbb{R}_+, E^n)} &\le \frac{N}{\nu} e^{\nu r} \sup_{t \ge 0} |F(t, \psi_{1t}) - F(t, \psi_{2t})| \\ &\le \frac{N}{\nu} e^{\nu r} L \sup_{t \ge 0} ||\psi_{1t} - \psi_{2t}|| = \frac{N}{\nu} e^{\nu r} L ||\psi_1 - \psi_2||_{AP(\mathbb{R}_+, E^n)} \end{aligned} \tag{5.24}$$

and hence

$$||\Phi(\psi_1) - \Phi(\psi_2)||_{AP(\mathbb{R}_+, E^n)} \le N\nu^{-1} e^{\nu r} L ||\psi_1 - \psi_2||_{AP(\mathbb{R}_+, E^n)}. \tag{5.25}$$

So, the mapping Φ has a unique fixed point $\varphi \in AP(\mathbb{R}_+, E^n)$, which is the desired solution. The theorem is proved. □

Corollary 5.16. *Let $A \in C(\mathbb{R}, \mathfrak{D})$ and $f \in C(\mathbb{R}, E^n)$ be asymptotically almost periodic and the zero solution of* (5.11) *be uniformly exponentially stable. If the mapping $F \in C(\mathbb{R} \times \mathfrak{C}, E^n)$ is asymptotically almost periodic with respect to $t \in \mathbb{R}$ uniformly with respect to φ on compacts from $\mathfrak{C}$ and satisfies the condition of Lipschitz with respect to the second argument, then there exists $\varepsilon_0 > 0$ such that for every ε, $|\varepsilon| \le \varepsilon_0$, the equation*

$$\frac{dx}{dt}(t) = A(t)x_t + f(t) + \varepsilon F(t, x_t) \tag{5.26}$$

has at least one asymptotically almost periodic solution φ_ε, and $\varphi_\varepsilon \to \varphi_0$ as $\varepsilon \to 0$ in the space $AP(\mathbb{R}_+, E^n)$, where φ_0 is a unique asymptotically almost periodic solution of (5.12) *satisfying the initial condition $\varphi_0(0) = 0$.*

Theorem 5.4.3. *Let $f \in C(\mathbb{R} \times \mathfrak{C}, E^n)$ be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent) with respect to $t \in \mathbb{R}$ uniformly with respect to φ on compact subsets from $\mathfrak{C}$. If there exists $\alpha > 0$ such that*

$$\operatorname{Re}\langle \varphi_1(0) - \varphi_2(0), f(t, \varphi_1) - f(t, \varphi_2)\rangle \le -\alpha |\varphi_1(0) - \varphi_2(0)|^2 \tag{5.27}$$

for all $t \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in C$, then (5.3) *is convergent.*

Proof. The formulated statement is proved by the same scheme that Theorem 3.8.5, that is why we omit its proof. □

5.5. Integral Equations of Volterra and Generated by the Nonautonomous Dynamical Systems

5.5.1. Nonlinear Integral Equations of Volterra

Let $(C(\mathbb{R}, E^n), \mathbb{R}, \sigma)$ be the dynamical system of shifts in the space $C(\mathbb{R}, E^n)$ of continuous on $\mathbb{R}$ functions with values in E^n with the compact-open topology. If on $\mathbb{R}^2 \times E^n$ we define a dynamical system by the rule $\pi(\tau, ((t,s), x)) := ((t+\tau, s+\tau), x)$, then according to Corollary 1.44 on the space $C(\mathbb{R}^2 \times E^n, E^n)$ of all continuous functions $f : \mathbb{R}^2 \times E^n \to E^n$ with the compact-open topology naturally there is defined a dynamical system of shifts $(C(\mathbb{R}^2 \times E^n, E^n), \mathbb{R}, \sigma)$. Assume $C_0(\mathbb{R}^2 \times E^n, E^n) := \{f \mid f \in C(\mathbb{R}^2 \times E^n, E^n),\ f(t,s,x) = 0$ for all $s \geq t\ x \in E^n\}$ and note that $C_0(\mathbb{R}^2 \times E^n, E^n)$ is a closed invariant subset of $(C(\mathbb{R}^2 \times E^n, E^n), \mathbb{R}, \sigma)$ and, consequently, on $C_0(\mathbb{R}^2 \times E^n, E^n)$ there is defined a dynamical system of shifts $(C_0(\mathbb{R}^2 \times E^n, E^n), \mathbb{R}, \sigma)$.

Let us consider an integral equation

$$x(t) = f(t) + \int_0^t F(t, s, x(s))\,ds, \tag{5.28}$$

where $f \in C(\mathbb{R}, E^n)$ and $F \in C_0(\mathbb{R}^2 \times E^n, E^n)$. Denote $H(F) := \overline{\{F^{(\tau)} : \tau \in \mathbb{R}\}}$, where $F^{(\tau)}(t,s,x) = F(t+\tau, s+\tau, x)$ and by bar there is denoted the closure in $C_0(\mathbb{R}^2 \times E^n, E^n)$.

Definition 5.17. The function $F \in C_0(\mathbb{R}^2 \times E^n, E^n)$ is called regular, if for any $G \in H(F)$ and $g \in C(\mathbb{R}, E^n)$ the equation

$$y(t) = g(t) + \int_0^t G(t, s, y(s))\,ds \tag{5.29}$$

has a unique solution.

Everywhere in this chapter we will consider only (5.28) with the regular right-hand side F.

From (5.28) it follows that

$$x(t+\tau) = f(t+\tau) + \int_0^\tau F(t+\tau, s, x(s))\,ds + \int_0^t F(t+\tau, s+\tau, x(s+\tau))\,ds. \tag{5.30}$$

Denote by $\varphi(t, f, F)$ the unique solution of (5.28). Then from general properties of the integral equations of Volterra [148] it follows that the mapping $\varphi : \mathbb{R} \times C(\mathbb{R}, E^n) \times C_0(\mathbb{R}^2 \times E^n, E^n) \to E^n$ is continuous. Let us define a mapping $\mathcal{F} : \mathbb{R} \times C(\mathbb{R}, E^n) \times C_0(\mathbb{R}^2 \times E^n, E^n) \to E^n$ by the equality

$$\mathcal{F}(\tau, \varphi, F)(t) := \int_0^\tau F(t+\tau, s, \varphi(s))\,ds \tag{5.31}$$

and a mapping

$$T : \mathbb{R} \times C(\mathbb{R}, E^n) \times C_0(\mathbb{R}^2 \times E^n, E^n) \longrightarrow C(\mathbb{R}, E^n) \tag{5.32}$$

by the rule

$$T(\tau, f, F) := f^{(\tau)} + \mathcal{F}(\tau, \varphi(\cdot, f, F), F). \tag{5.33}$$

From equality (5.30) it follows that

$$\varphi(t+\tau, f, F) = \varphi(t, T(\tau, f, F), F^{(\tau)}) \tag{5.34}$$

for all $f \in C(\mathbb{R}, E^n)$ and $t, \tau \in \mathbb{R}$, and besides

$$\varphi(\tau, f, F) = T(\tau, f, F)(0). \tag{5.35}$$

The definition of T directly implies the equality

$$T(t+\tau, f, F) = T(t, T(\tau, f, F), F^{(\tau)}) \tag{5.36}$$

for all $t, \tau \in \mathbb{R}$ and $f \in C(\mathbb{R}, E^n)$.

Example 5.18. Put $Y := H(F)$ and let $(Y, \mathbb{R}, \sigma)$ be a dynamical system of shifts on Y. Denote $X := C(\mathbb{R}, E^n) \times Y$ and define a mapping $\pi : X \times \mathbb{R} \to X$ by the following rule: $\pi(\tau, (g, G)) := (T(\tau, g, G), G^{\tau})$ for all $(g, G) \in X := C(\mathbb{R}_+, E^n) \times H(F)$ and $\tau \in \mathbb{R}_+$. From the above said it follows that the triplet $(X, \mathbb{R}_+, \pi)$ is a dynamical system (more detailed about that see in [148]). Assume $h := pr : X \to Y$. Then the triplet $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a nonautonomous dynamical system generated by (5.28).

Let $(X, \mathbb{R}_+, \pi)$ be a dynamical system on $X = C(\mathbb{R}, E^n) \times H(F)$ constructed in Example 5.18 and let us define a mapping $\lambda : \mathbb{R}_+ \times X \to E^n$ by the next rule: $\lambda(\tau, (g, G)) := \varphi(\tau, g, G)$. From equality (5.34) it follows that

$$\lambda(\tau, \pi(x, t)) = \lambda(t+\tau, x) \tag{5.37}$$

for all $t, \tau \in \mathbb{R}_+$ and $x \in X$.

Let $\lambda : \mathbb{T} \times X \to Y$ be a continuous mapping.

Definition 5.19. One will say that the family of mappings $\{\lambda(t, \cdot) : t \in \mathbb{T}\}$ from $X \to Y$ separates points, if for every two different points $x_1, x_2 \in X$ there exists $t = t(x_1, x_2) \in \mathbb{T}$ such that $\lambda(t, x_1) \neq \lambda(t, x_2)$.

There takes place the following lemma.

Lemma 5.20. *Let $(X, \mathbb{T}, \pi)$ be a dynamical system, Y be a full metric space, and $\lambda : \mathbb{T} \times X \to Y$ be a continuous mapping satisfying condition* (5.37) *and $(C(\mathbb{T}, Y), \mathbb{T}, \sigma)$ be a dynamical system of shifts on $C(\mathbb{T}, Y)$. If the family of mappings $\{\lambda(t, \cdot) : t \in \mathbb{T}\}$ separates points, then the mapping $p : X \to C(\mathbb{T}, Y)$ defined by the equality $p(x) := \varphi_x \in C(\mathbb{T}, Y)$, where $\varphi_x(t) := \lambda(t, x)$ for all $t \in \mathbb{T}$, is a homeomorphism of $(X, \mathbb{T}, \pi)$ onto $(p(X), \mathbb{T}, \sigma)$, that is,*

(1) *h is continuous, one-to-one and $h^{-1} : p(X) \to X$ also is continuous;*
(2) *$p(\pi(t, x)) = \sigma(t, p(x))$ for all $t \in \mathbb{T}$ and $x \in X$.*

Proof. Let us show that the mapping h is continuous. Let $x_k \to x_0$. We will show that $p(x_k) \to p(x_0)$. Suppose that it is not so. Then there are a number $\varepsilon_0 > 0$, a compact set $K_0 \subset \mathbb{T}$, and a subsequence m_k such that

$$\max_{t \in K_0} \rho\big(\varphi_{x_{m_k}}(t), \varphi_x(t)\big) \geq \varepsilon_0. \tag{5.38}$$

Then there exists a subsequence $\{t_k\} \subset K$ such that

$$\rho\big(\lambda(t_k, x_{m_k}), \lambda(t_k, x_0)\big) \geq \varepsilon_0. \tag{5.39}$$

By the compactness of K_0 the sequence $\{t_k\}$ can be considered convergent. Assume $t_0 = \lim_{k\to\infty} t_k$ and in inequality (5.38) let us pass to the limit as $k \to +\infty$. Then we get $\varepsilon_0 \leq 0$, and that contradicts to the choice of ε_0. The obtained contradiction proves the continuity of p.

The fact that the family $\{\lambda(t, \cdot) : t \in \pi\}$ separates points imply that the mapping p is one-to-one. Obviously, $p^{-1} : p(X) \to X$ is continuous.

At last, note that

$$p(\pi(t,x))(s) = \lambda(s, \pi(t,x)) = \lambda(t+s, x) = \varphi_x(t+s) = \sigma(t, \varphi_x)(s), \tag{5.40}$$

that is, $p(\pi(t,x)) = \sigma(t, p(x))$ for all $x \in X$ and $t \in \mathbb{T}$. The lemma is proved. □

Corollary 5.21. *The dynamical system* $(X, \mathbb{R}_+, \pi)$ *constructed in Example 5.18 is homeomorphically embedded in the dynamical system of shifts* $(C(\mathbb{R}, E^n), \mathbb{R}_+, \sigma)$.

Example 5.22. Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ be the nonautonomous dynamical system constructed in Example 5.18. According to Corollary 5.21 there exists a homeomorphism p of the dynamical system $(X, \mathbb{R}_+, \pi)$ onto $(W, \mathbb{R}_+, \sigma)$, where $W = p(X)$. Assume $q := h \circ p : W \to Y$. Then $\langle (W, \mathbb{R}_+, \sigma)$, $(Y, \mathbb{R}_+, \sigma), q\rangle$ also is a nonautonomous dynamical system associated by (5.28).

5.5.2. Linear Integral Equations

Let $(C(\mathbb{R}^2, [E^n]), \mathbb{R}, \sigma)$ be the dynamical system of shifts on the space $C(\mathbb{R}^2, [E^n])$ of al the continuous matrix-functions $A : \mathbb{R}^2 \to [E^n]$ with compact-open topology, that is, $\sigma(\tau, A) = A^{(\tau)}$ and $A^{(\tau)}(t,s) := A(t+\tau, s+\tau)$. By $C_0(\mathbb{R}^2, [E^n])$ we denote the set of all $A \in C(\mathbb{R}^2, [E^n])$ satisfying the condition $A(t,s) = 0$ for all $s \geq t$. It is clear that $C_0(\mathbb{R}^2, [E^n])$ is a closed and invariant set in the dynamical system of shifts $(C(\mathbb{R}^2, [E^n]), \mathbb{R}, \sigma)$. Hence, on $C_0(\mathbb{R}^2, [E^n])$ there is induced a dynamical system system of shifts $(C_0(\mathbb{R}^2, [E^n]), \mathbb{R}, \sigma)$.

Let us consider a linear integral equation

$$x(t) = f(t) + \int_0^t A(t,s) x(s) ds, \tag{5.41}$$

where $f \in C(\mathbb{R}, E^n)$ and $A \in C_0(\mathbb{R}^2, [E^n])$. Denote by $\varphi(t, f, A)$ the unique solution of

(5.41) defined on $\mathbb{R}$. Then

$$\varphi(t+\tau, f, A) = T(\tau, f, A)(t) + \int_0^t A^{(\tau)}(t,s)\varphi(s+\tau, f, A)ds, \tag{5.42}$$

where $T(\tau, f, A)(t) := f^{(\tau)}(t) + \int_0^t A(t+\tau, s)\varphi(s, f, A)ds$.

Example 5.23. Let $Y = H(A) := \overline{\{A^{(\tau)} \mid \tau \in \mathbb{R}\}}$ and $(Y, \mathbb{R}, \sigma)$ be a dynamical system of shifts. Put $X := C(\mathbb{R}, E^n) \times H(A)$ and define a dynamical system $(X, \mathbb{R}, \pi)$ by the following rule: $\pi(\tau, (f, A)) = (T(\tau, f, A), A^{(\tau)})$. Then $\langle (X, \mathbb{R}, \pi), (Y, \mathbb{R}, \sigma), h \rangle$, where $h := pr_2 : X \to Y$, is a linear nonautonomous dynamical system generated by (5.41).

5.6. Asymptotically Almost Periodic Solutions for Integral Equations of Volterra

For integral equations of Volterra, as well as for ordinary differential equations and FDEs, we can obtain series of tests of asymptotical almost periodicity, if we apply the results of Chapter 2 to the dynamical systems from Examples 5.18, 5.22, and 5.23. Before formulating the according statements we will do the following.

Remark 5.24. (a) Let $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$ and $\varphi(t, f, F)$ be a solution of (5.28) such that $\{\varphi(t+\tau, f, F) \mid \tau \in \mathbb{T}\} \subset C(\mathbb{R}, E^n)$ is relatively compact. If $\{F^{(\tau)} \mid \tau \in \mathbb{T}\} \subset C(\mathbb{R}^2 \times E^n, E^n)$ is relatively compact, then $\{T(\tau, f, F) : \tau \in \mathbb{T}\}$ also is relatively compact in $C(\mathbb{R}, E^n)$.

(b) Let $A \in C(\mathbb{R}^2, [E^n])$ and $\{A^{(\tau)} \mid \tau \in \mathbb{T}\}$ be relatively compact. If $\varphi(t, f, A)$ is a solution of (5.39) such that $\{\varphi(t+\tau, f, A) : \tau \in \mathbb{T}\}$ is relatively compact, then $\{T(\tau, f, A) \mid \tau \in \mathbb{T}\}$ is relatively compact.

Theorem 5.6.1. *Let $F \in C_0(\mathbb{R}^2 \times E^n, E^n)$ be asymptotically almost periodic (i.e., the motion $\sigma(\cdot, F)$ of the dynamical system $(C_0(\mathbb{R}^2 \times E^n, E^n), \mathbb{R}, \sigma)$ is asymptotically almost periodic) and $\varphi(t, f, F)$ is a solution of* (5.28) *such that the set $\{\varphi(t+\tau, f, F) \mid \tau \in \mathbb{R}_+\}$ is relatively compact in $C(\mathbb{R}, E^n)$. If for every $G \in \omega_F = \{G \mid G \in C_0(\mathbb{R}^2 \times E^n, E^n),\ \exists t_n \to +\infty$ such that $F^{(\tau_n)} \to G\}$ and $g \in C(\mathbb{R}, E^n)$* (5.29) *has at most one solution from ω_φ, then the solution φ is asymptotically almost periodic.*

Theorem 5.6.2. *Let $A \in C(\mathbb{R}, [E^n])$ and $f \in C(\mathbb{R}, E^n)$ be asymptotically almost periodic, $B \in C(\mathbb{R}^2, [E^n])$ be asymptotically almost periodic (i.e., the motion $\sigma(\tau, B)$ is asymptotically almost periodic in $(C(\mathbb{R}^2, [E^n]), \mathbb{R}, \sigma)$). If φ is a solution of the equation*

$$\frac{dx}{dt}(t) = A(t)x(t) + f(t) + \int_0^t B(t,s)x(s)ds \tag{5.43}$$

such that $\{\varphi(t+\tau) \mid \tau \in \mathbb{R}_+\} \subset C(\mathbb{R}, E^n)$ is relatively compact and for every $\widetilde{A} \in \omega_A$, $\widetilde{f} \in \omega_f$ and $\widetilde{B} \in \omega_B$ the equation

$$\frac{dy(t)}{dt} = \widehat{A}(t)y(t) + \widetilde{f}(t) + \int_0^t \widetilde{B}(t,s)y(s)ds \tag{5.44}$$

has at most one solution from ω_φ, then φ is asymptotically almost periodic.

Remark 5.25. Every solution $\varphi(t,x_0,f,A,B)$ of (5.43) satisfies the integral equation

$$x(t) = \hat{f}(t) + \int_0^t \hat{A}(t,s)x(s)ds, \tag{5.45}$$

where $\hat{f}(t) := x_0 + \int_0^t f(s)ds$ and $\hat{A}(t,s) := A(s) + \int_s^t B(u,s)du$.

Consider the integral equation of Volterra of convolutional type

$$x(t) = f(t) + \int_0^t A(t,s)x(s)ds, \tag{5.46}$$

where $f \in C(\mathbb{R},E^n)$ and $A \in C(\mathbb{R},[E^n])$.

Definition 5.26. A resolvent of integral equation (5.46) is called [149] a matrix-function $\mathcal{R} \in C(\mathbb{R},[E^n])$ satisfying the equation

$$\mathcal{R}(t) = -A(t) + \int_0^t A(t-s)\mathcal{R}(s)ds. \tag{5.47}$$

The solution of (5.46) is given by the formula

$$x(t) = f(t) - \int_0^t \mathcal{R}(t-s)f(s)ds, \tag{5.48}$$

where $\mathcal{R}$ is the resolvent of (5.46).

Definition 5.27. They say [149] that the resolvent $\mathcal{R}$ of (5.46) is hyperbolic (satisfies the condition of exponential dichotomy on $\mathbb{R}$), if there exist a pair of jointly complimentary projectors P_1 and P_2 and positive numbers N and ν such that

$$\begin{aligned} ||\mathcal{R}(t)P_1|| &\le Ne^{+\nu t} \quad (t \in \mathbb{R}_-), \\ ||\mathcal{R}(t)P_2|| &\le Ne^{-\nu t} \quad (t \in \mathbb{R}_+). \end{aligned} \tag{5.49}$$

Theorem 5.6.3. *Let $f \in C(\mathbb{R},E^n)$ be bounded on $\mathbb{R}$, $A \in C(\mathbb{R},[E^n])$ the resolvent $\mathcal{R}(t)$ of (5.47) be hyperbolic on $\mathbb{R}$. Then the solution φ of (5.46) is uniformly compatible in limit, that is, $\mathfrak{L}_f \subseteq \mathfrak{L}_\varphi$.*

Proof. Let us introduce in consideration two operators L and B by the following rules:

$$\begin{aligned} (Lf)(t) &:= \int_{-\infty}^0 \mathcal{R}(t-s)P_2f(s)ds - \int_0^{+\infty} \mathcal{R}(t-s)P_1f(s)ds, \\ (Nf)(t) &:= \int_{-\infty}^t \mathcal{R}(t-s)P_2f(s)ds - \int_0^{+\infty} \mathcal{R}(t-s)P_1f(s)ds. \end{aligned} \tag{5.50}$$

According to [149] equality (5.48) can be rewritten as follows:

$$x = f - Lf + Nf, \tag{5.51}$$

and, consequently, $x+Lf = f+Nf$. Assuming $y := x+Lf = f+Nf$, we can show [149] that y is a solution of the integral equation

$$y(t) = f^*(t) + \int_0^t A(t-s)y(s)ds, \tag{5.52}$$

where $f^* = (I-A)(I+N)f$.

Let us show now that $\mathfrak{L}_f \subseteq \mathfrak{L}_y$. Let $\{\tau_k\} \in \mathfrak{L}_f$. Then $|\tau_n| \to +\infty$ and there exists $g \in C(\mathbb{R}, E^n)$ such that $f^{(\tau_k)} \to g$ is uniform on compact subsets from $\mathbb{R}$. Since $y = f+Nf$, it is enough to show that $(Nf)^{(\tau_k)} \to Ng$ is uniform on compact subsets from $\mathbb{R}$. Let $K \subset \mathbb{R}$ be some compact set and $t \in K$. Since $(Nf)^{(\tau)} = N(f^{(\tau)})$,

$$\begin{aligned} &|(Nf)(t+\tau_k) - (Ng)(t)| \\ &\quad \le \left| \int_{-\infty}^t \mathcal{R}(t-s)P_2[f(s+\tau_n) - g(s)]ds \right| + \left| \int_t^{+\infty} \mathcal{R}(t-s)P_1[f(s+\tau_k) - g(s)]ds \right|. \end{aligned} \tag{5.53}$$

Let us show that

$$\sup_{t \in K} \left| \int_{-\infty}^t \mathcal{R}(t-s)P_2[f(s+\tau_k) - g(s)]ds \right| \longrightarrow 0, \tag{5.54}$$

as $k \to +\infty$. Let $\varepsilon > 0$. Since the integral

$$\int_{-\infty}^t \mathcal{R}(t-s)P_2[f(s+\tau_k) - g(s)]ds = \int_0^{+\infty} \mathcal{R}(u)P_2[f(t-u+\tau_k) - g(t-u)]du \tag{5.55}$$

is absolutely convergent uniformly with respect to k, there exist a number $L = L(\varepsilon) > 0$ such that

$$\left| \int_L^{+\infty} \mathcal{R}(u)P_2[f(t-u+\tau_k) - g(t-u)]d\tau \right| < \frac{\varepsilon}{4} \tag{5.56}$$

for all $k \in \mathbb{N}$ and $t \in K$. On the other hand,

$$\begin{aligned} &\left| \int_0^L \mathcal{R}(u)P_2[f(t-u+\tau_k) - g(t-u)]du \right| \\ &\qquad \le \sup_{0 \le u \le L} |f(t-u+\tau_k) - g(t-u)| \int_0^L ||\mathcal{R}(u)P_2||du \\ &\qquad \le \sup_{s \in K'} |f(s+\tau_k) - g(s)| \int_0^{+\infty} ||\mathcal{R}(u)P_2||du \frac{N}{\nu} \sup_{s \in K'} |f(s+\tau_k) - g(s)|, \end{aligned} \tag{5.57}$$

where $K' = \{t-u \mid t \in K, u \in [0,L]\}$ is some compact subset from $\mathbb{R}$. As $f_{\tau_k} \to g$, then there is $k_1(\varepsilon) > 0$ such that

$$\sup_{s \in K_1} |f(s+\tau_k) - g(s)| < \frac{(\nu\varepsilon)}{4N} \tag{5.58}$$

for all $k \geq k_1(\varepsilon)$. From inequalities (5.56) and (5.58) it follows that

$$\sup_{t\in K}\left|\int_{-\infty}^{t}\mathcal{R}(t-s)P_2[f(s+\tau_k)-g(s)]ds\right|<\frac{\varepsilon}{2} \tag{5.59}$$

for all $k \geq k_1(\varepsilon)$.

On the analogy, there exists $k_2(\varepsilon) > 0$ such that

$$\sup_{t\in K}\left|\int_{t}^{+\infty}\mathcal{R}(t-s)P_1[f(s+\tau_k)-g(s)]ds\right|<\frac{\varepsilon}{2} \tag{5.60}$$

for all $k \geq k_2(\varepsilon)$. Put $k(\varepsilon) := \max(k_1(\varepsilon), k_2(\varepsilon))$. Then from (5.53), (5.59), and (5.60) it follows that

$$\sup_{t\in K}\left|(Nf)(t+\tau_k)-(Ng)(t)\right|<\varepsilon \tag{5.61}$$

for all $k \geq k(\varepsilon)$. The theorem is proved. □

Corollary 5.28. *Let the resolvent $\mathcal{R}(t)$ of* (5.47) *be hyperbolic on $\mathbb{R}$. Then the next statements take place.*

(1) *If f is bilaterally asymptotically stationary (resp., bilaterally asymptotically periodic, bilaterally asymptotically almost periodic, bilaterally asymptotically recurrent), then the solution φ of* (5.47) *possesses this property too.*
(2) *If f is stationary (resp., periodic, almost periodic, recurrent) homoclinic, then φ also is.*

Proof. This statement follows from Theorem 5.6.3 and Corollary 2.23. □

5.7. Convergence of Some Evolution Equations

(1) Let $\mathcal{H}$ be a real Hilbert space, $D(A) \subseteq \mathcal{H}$ be the domain of definition of the operator $A : \mathcal{H} \to \mathcal{H}$.

Definition 5.29. Recall [107, 150] that the operator A is called

(1) monotone, if for every $u_1, u_2 \in D(A) : \langle Au_1 - Au_2, u_1 - u_2\rangle \geq 0$;
(2) semicontinuous, if the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by the equality $\varphi(\lambda) := \langle A(u + \lambda v, w)\rangle$ is continuous;
(3) uniformly monotone, if there exists a positive number α such that $\langle Au - Av, u - v\rangle \geq \alpha|u - v|^2$ for all $u, v \in D(A)$ ($|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$ and $\langle\cdot,\cdot\rangle$ is a scalar product in $\mathcal{H}$).

Note that the family of monotone operators can be partially ordered by including graphics.

Definition 5.30. A monotone operator is called maximal, if it is maximal among the monotone operators.

Let us consider an evolutionary equation

$$\frac{dx}{dt} + Ax = f(t), \tag{5.62}$$

where $f \in L^1_{\text{loc}}(\mathbb{R}, \mathcal{H})$ and A is a maximal monotone operator with the domain of definition $D(A)$. According to [150] for every $x_0 \in \overline{D(A)}$ there exists a unique weak solution $\varphi(t, x_0, f)$ of (5.62) satisfying the condition $\varphi(0, x_0, f) = x_0$ and defined on $\mathbb{R}_+$. Let $Y := H(f) = \overline{\{f^{(\tau)} \mid \tau \in \mathbb{R}\}}$, where by bar it is denoted the closure in $L_1(\mathbb{R}, \mathcal{H})$. By $(Y, \mathbb{R}, \sigma)$ we denote the dynamical system of shifts on Y induced by the dynamical system $(L^1_{\text{loc}}(\mathbb{R}, \mathcal{H}), \mathbb{R}, \sigma)$. Put $X := \overline{D(A)} \times Y$ and define $\pi : \mathbb{R}_+ \times \overline{D(A)} \times Y \to \overline{D(A)} \times Y$ by the equality $\pi(t, (v, g)) := (\varphi(t, v, g), g^{(t)})$ and $h := pr_2 : X \to Y$. As it is shown in the work [39], the triplet $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ is a nonautonomous dynamical system. Applying to the constructed nonautonomous dynamical systems the results of Chapter 2 we obtain the corresponding statements for (5.62). Let us give one statement of this kind.

Theorem 5.7.1. *Let a mapping $f \in L^1_{\text{loc}}(\mathbb{R}, \mathcal{H})$ be asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent). If the maximal monotone operator A is semicontinuous and uniformly monotone, then* (5.62) *is convergent.*

Proof. The proof is executed by the same scheme that the proof of Theorem 3.8.5, taking into account the fact that according to the results of work [39] from the uniform monotonicity of the operator A it follows the existence of positive numbers N and ν such that

$$|\varphi(t, u_1, g) - \varphi(t, u_2, g)| \leq N e^{-\nu t} |u_1 - u_2| \tag{5.63}$$

for all $u_1, u_2 \in \overline{D(A)}$ and $g \in H(f)$. □

Note that in the almost periodic case Theorem 5.7.1 revises and reinforces one result from [107, page 164].

Let us give an example of the equation of type (5.62).

Consider an equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u - \phi\left(\frac{\partial u}{\partial t}\right) + f(t) \tag{5.64}$$

in the open bounded area $\Omega \subset E^n$ with the boundary condition $u = 0$ on $\partial\Omega$. Suppose that the function $\phi : \mathbb{R} \to \mathbb{R}$ satisfies the conditions $\phi(0) = 0$ and $0 < c_1 \leq \phi'(\xi) \leq c_2$ $(\xi \in \mathbb{R})$. Then (5.64) will be rewritten in the form

$$\begin{aligned} \partial_t u &= v, \\ \partial_t v &= \triangle u - \phi(v) + f(t). \end{aligned} \tag{5.65}$$

At last, put $\mathcal{H} = W_0^{1,2}(\Omega) \times L^2(\Omega)$ and define on $\mathcal{H}$ a scalar product

$$\langle (u, v), (u^*, v^*) \rangle = \int_\Omega [v v^* + \nabla u \nabla u^* + \lambda u v^* + \lambda u^* v] dx, \tag{5.66}$$

where λ is some positive constant depending only on c_1 and c_2. We can check (see, i.e., [151]) that under assumptions made all the conditions of Theorem 5.7.1 are fulfilled, if $f \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R})$ is asymptotically stationary (resp., asymptotically τ-periodic, asymptotically almost periodic, asymptotically recurrent).

(2) Let $\mathfrak{B}$ be a Banach space, $I \subseteq \mathbb{R}$ and $\mathcal{D}(\mathbb{R}, \mathfrak{B})$ be the space of all the infinitely differentiable finite functions $\varphi : \mathbb{R} \to \mathfrak{B}$ and $\mathcal{H}$ be a complex Hilbert space.

Consider the equation

$$\int_{\mathbb{R}} [\langle u(t), \varphi'(t)\rangle + \langle A(t)u(t), \varphi(t)\rangle + \langle f(t), \varphi(t)\rangle]dt = 0, \tag{5.67}$$

where $A \in C(\mathbb{R}, [\mathcal{H}])$, $f \in C(\mathbb{R}, \mathcal{H})$ and $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathcal{H}$. The function $\varphi \in C(I, \mathcal{H})$ is called a solution of (5.67), if equality (5.67) takes place for every $\varphi \in \mathcal{D}(I, \mathcal{H})$. Let $x \in \mathcal{H}$ and $\varphi(t, x, A, f)$ be a solution of (5.67) defined on $\mathbb{R}_+$ and satisfying to the condition $\varphi(0, x, A, f) = x$.

Denote by $(C(\mathbb{R}, [\mathcal{H}]), \mathbb{R}, \sigma)$ and $(C(\mathbb{R}, \mathcal{H}), \mathbb{R}, \sigma)$ dynamical systems of shifts on $C(\mathbb{R}, [\mathcal{H}])$ and $C(\mathbb{R}, \mathcal{H})$, respectively, and let $(C(\mathbb{R}, [\mathcal{H}]) \times C(\mathbb{R}, \mathcal{H}), \mathbb{R}, \sigma)$ be their product. Put $H(A, f) := \overline{\{(A^{(\tau)}, f^{(\tau)}) : \tau \in \mathbb{R}\}}$ and let $(H(A, f), \mathbb{R}, \sigma)$ be a dynamical system of shifts on $H(A, f)$. Along with (5.67) we will consider the family of equations

$$\int_{\mathbb{R}} [\langle u(t), \varphi'(t)\rangle + \langle B(t)u(t), \varphi(t)\rangle + \langle g(t), \varphi(t)\rangle]dt = 0, \tag{5.68}$$

where $(B, g) \in H(A, f)$.

Everywhere in this paragraph we will suppose that the operator function $A(t)$ is self-adjoint and negatively defined, that is, $A(t) = -A_1(t) + iA_2(t)$ for all $t \in \mathbb{R}$, where $A_1(t)$ and $A_2(t)$ are self-adjoint operators and

$$\langle A_1(t)x, x\rangle \geq \alpha |x|^2 \tag{5.69}$$

for all $x \in \mathcal{H}$, $t \in \mathbb{R}$, $|\cdot|^2 = \langle \cdot, \cdot \rangle$, and $\alpha > 0$.

Lemma 5.31 (see [5]). *For all $t > 0$ there takes place the equality*

$$\frac{1}{2}\frac{d}{dt}|\varphi(t, x, A, f)|^2 = -\langle A_1(t)\varphi(t, x, A, f), \varphi(t, x, A, f)\rangle + \operatorname{Re}\langle f(t), \varphi(t, x, A, f)\rangle. \tag{5.70}$$

Lemma 5.32. *For all $t \in \mathbb{R}_+$ there takes place the inequality*

$$|\varphi(t, x, A, f)| \leq |x| + \int_0^t |f(\tau)| d\tau. \tag{5.71}$$

Proof. From equality (5.70) it follows that

$$\frac{1}{2}\frac{d}{dt}|\varphi(t, x, A, f)|^2 \leq |f(t)|\,|\varphi(t, x, A, f)|. \tag{5.72}$$

Assume $v(t) := |\varphi(t, x, A, f)|^2$. Then $dv/dt \leq 2|f(t)|\sqrt{v(t)}$ and, consequently, $\sqrt{v(t)} - \sqrt{v(\tau)} \leq \int_\tau^t |f(s)| ds$. Hence, $|\varphi(t, x, A, f)| \leq |x| + \int_0^t |f(\tau)| d\tau$. The lemma is proved. □

Lemma 5.33. *Let l, r, and $\beta > 0$, $x_0 \in \mathcal{H}$, $A \in C(\mathbb{R}, [\mathcal{H}])$ and $f \in C(\mathbb{R}, \mathcal{H})$. Then there exists a number $M = M(f, l, r, \beta, x_0) > 0$ such that*

$$\begin{aligned} &|\varphi(t,x,B,g) - \varphi(t,x_0,A,f)| \\ &\qquad \leq |x - x_0| + M\int_0^t |B(\tau) - A(\tau)|\,d\tau + \int_0^t |g(\tau) - f(\tau)|\,d\tau \end{aligned} \tag{5.73}$$

for all $t \in [0,l]$ and $x \in B[x_0, r_0]$, if $|g(t) - f(t)| \leq \beta$ and $\operatorname{Re}\langle B(t)x, x\rangle \leq 0$ for all $t \in [0,l]$ and $x \in \mathcal{H}$.

Proof. Let $v(t) = [\varphi(t,x,B,g) - \varphi(t,x_0,A,f)]$. Then

$$\int_{\mathbb{R}} \{\langle v(t), \varphi'(t)\rangle + \langle A(t)v(t), \varphi(t)\rangle + \langle (B(t) - A(t))v(t), \varphi(t)\rangle + \langle g(t) - f(t), \varphi(t)\rangle\}\,dt = 0 \tag{5.74}$$

for every $\varphi \in \mathcal{D}(\mathbb{R}, \mathcal{H})$. According to Lemma 5.31

$$\begin{aligned} \frac{d}{2dt}|v(t)|^2 &= \operatorname{Re}\langle A(t)v(t), v(t)\rangle \\ &\quad + \operatorname{Re}[\langle (B(t) - A(t))\varphi(t,x,B,g), v(t)\rangle + \langle g(t) - f(t), v(t)\rangle], \end{aligned} \tag{5.75}$$

and from Lemma 5.32 we have

$$\begin{aligned} |v(t)| &\leq |v(0)| + \int_0^t |(B(\tau) - A(\tau))v(\tau) + g(\tau) - f(\tau)|\,d\tau \\ &\leq |v(0)| + \int_0^t |B(\tau) - A(\tau)|\,|\varphi(\tau,x,B,g)|\,d\tau + \int_0^t |g(\tau) - f(\tau)|\,d\tau. \end{aligned} \tag{5.76}$$

On the other hand, according to Lemma 5.32 for $\varphi(\tau,x,B,g)$ we have

$$|\varphi(t,x,B,g)| \leq |x| + \int_0^t |g(\tau)|\,d\tau \leq |x_0| + r + \beta l + l\max_{0\leq t\leq l}|f(t)| = M(f,l,r,\beta,x_0). \tag{5.77}$$

From inequalities (5.76) and (5.77) it follows (5.73). Lemma is proved. □

Put $\widetilde{\mathcal{H}} := \mathcal{H} \times H(A,f)$ and by X denote the set of all the pairs $(u,(B,g))$ from $\mathcal{H} \times H(A,f)$ such that through the point $x \in \mathcal{H}$ as $t = 0$ there passes a solution $\varphi(t,u,B,g)$ of (5.68) defined on $\mathbb{R}_+$.

Lemma 5.34. *The set X is closed in $\mathcal{H} \times H(A,f)$.*

Proof. Let $(x,(A,f)) \in \overline{X}$. Then there exists a sequence $\langle x_k,(B_k,g_k)\rangle \in X$ such that $x_k \to x$ in $\mathcal{H}$, $B_k \to A$ in $C(\mathbb{R},[\mathcal{H}])$ and $f_k \to f$ in $C(\mathbb{R},\mathcal{H})$. Let $l, \varepsilon > 0$. Then there exists $k_0 = k_0(\varepsilon, l) > 0$ such that

$$|x_k - x_l| < \varepsilon,\ |f_k(t) - f_l(t)| < \varepsilon\ |B_k(t) - B_l(t)| < \varepsilon \tag{5.78}$$

for all $t \in [0,l]$ and $k,l \geq k_0$. Assume $r := \sup\{|x_k| \mid k \in \mathbb{N}\}$. Then by Lemma 5.32 we obtain

$$|\varphi(t,x_k,B_k,f_k) - \varphi(t,x_l,B_l,f_l)| \\ \leq |x_k - x_l| + M\int_0^t |B_k(\tau) - B_l(\tau)|\,d\tau + \int_0^t |f_k(\tau) - f_l(\tau)|\,d\tau \leq \varepsilon + M\varepsilon l + \varepsilon l \tag{5.79}$$

for all $t \in [0,l]$ and $k,l \geq k_0$, where M is some positive constant depending only on r, l and f. From (5.79) it follows that the sequence $\{\varphi(t,x_k,B_k,f_k)\}$ is fundamental in the space $C(\mathbb{R}_+,\mathcal{H})$ and, consequently, it is convergent in $C(\mathbb{R}_+,\mathcal{H})$. From (5.79) it follows that $\varphi(t,x_k,B_k,f_k) \to \varphi(t,x,A,f)$ in $C(\mathbb{R}_+,\mathcal{H})$ as $k \to +\infty$. So, $(x,A) \in X$, that is, $\overline{X} \subseteq X$. The lemma is proved. □

Lemma 5.35. *The mapping $\varphi : \mathbb{R}_+ \times X \to \mathcal{H}$ given by the rule $(t,(u,Bg)) \to \varphi(t,u,B,g)$ is continuous.*

Proof. Let $t_k \to t$, $x_k \to x$, $B_k \to B$ and $g_k \to g$. Then

$$|\varphi(t_k,x_k,B_k,g_k) - \varphi(t,x,B,g)| \\ \leq |\varphi(t_k,x_k,B_k,g_k) - \varphi(t_k,x,B,g)| + |\varphi(t_k,x,B,g) - \varphi(t,x,B,g)| \\ \leq \max_{0\leq t\leq l} |\varphi(t,x_k,B_k,g_k) - \varphi(t,x,B,g)| + |\varphi(t_k,x,B,g) - \varphi(t,x,B,g)|. \tag{5.80}$$

From (5.80) and Lemma 5.33 we get the necessary statement. The lemma is proved. □

Lemmas 5.34, 5.35, and general properties of solutions of the equations of the type (5.67) allow us to define on X a dynamical system $(X,\mathbb{R}_+,\pi)$ in the following way: $\pi(t,x) := \pi(t,(u,(B,g))) = (\varphi(t,u,B,g),B^{(t)},g^{(t)})$ for all $(u,(B,g)) \in X$ and $t \in \mathbb{R}_+$.

Put $Y := H(A,f)$ (resp., $Y := H^+(A,f)$). By $(Y,\mathbb{R},\sigma)$ (resp., $(Y,\mathbb{R}_+,\sigma)$) we denote a dynamical system (resp., a semigroup dynamical system) of shifts on Y. Let $h := pr_2 : X \to Y$. Then the triplet $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ (resp., $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R}_+,\sigma),h\rangle$) is a nonautonomous dynamical system generated by (5.67).

There takes place the following lemma.

Lemma 5.36. *For every $(B,g) \in H(A,f) = Y$ and $x_1,x_2 \in \mathcal{H}$ ($(x_i,B,g) \in X$, $i = 1,2$) there takes place the inequality*

$$|\varphi(t,x_1,B,g) - \varphi(t,x_2,B,g)| \leq e^{-\alpha t}|x_1 - x_2| \tag{5.81}$$

for all $t \in \mathbb{R}_+$, that is, the nonautonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ (resp., $\langle (X,\mathbb{R}_+,\pi)$, $(Y,\mathbb{R}_+,\sigma),h\rangle$) is contracting.

Proof. The formulated lemma directly follows from Lemma 5.31. In fact, assume $\omega(t) := \varphi(t,x_1,B,g) - \varphi(t,x_2,B,g)$. Then

$$\frac{1}{2}\frac{d}{dt}|\omega(t)|^2 = \operatorname{Re}\langle B(t)\omega(t),\omega(t)\rangle \leq -\alpha|\omega(t)|^2 \tag{5.82}$$

and, consequently, $|\omega(t)| \leq |\omega(0)|e^{-\alpha t}$ for all $t \in \mathbb{R}_+$. Lemma is proved. □

Theorem 5.7.2. *Let* $A \in C(\mathbb{R}, [\mathcal{H}])$ *and* $f \in C(\mathbb{R}, \mathcal{H})$ *be jointly asymptotically stationary (resp., jointly asymptotically* τ*-periodic, jointly asymptotically almost periodic, jointly asymptotically recurrent). Then* (5.67) *is convergent, that is, the nonautonomous dynamical system* $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ *generated by* (5.67) *is convergent.*

Proof. The formulated statement is proved in the same way that Theorem 3.8.5, using the above constructed nonautonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ and Lemma 5.35. □

Remark 5.37. Note, that in the case of asymptotical almost periodicity of $A(t)$ and $f(t)$ Theorem 5.7.2 reinforces one result from the work [5].

Following [5], we will give an example of the boundary problem reduced to a equation of type (5.67). Let Ω be a bounded domain in $\mathbb{R}^n$; $\Gamma := \partial\Omega$, $Q := \mathbb{R}_+ \times \Omega$, $S := \mathbb{R}_+ \times \Gamma$. In Q consider the first initial boundary problem for the equation

$$\frac{\partial u}{\partial t} = Lu + g(t, u), \ u|_{t=0} = \varphi(x), \ u|_S = 0. \tag{5.83}$$

Here $Lu := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(t)\frac{\partial u}{\partial x_j}) - a(t, x)u$ is a uniformly elliptic operator, that is, for every vector $\xi \in \mathbb{R}^n$

$$\lambda \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j \le \mu \sum_{i=1}^{n} \xi_i^2, \tag{5.84}$$

$\lambda > 0$. In virtue of the theorem of Riesz the operator $A(t)$ is defined from the condition

$$\langle A(t)u, \varphi\rangle := -\int_{\Omega}\left[\sum_{i,j=1}^{n} a_{ij}(t, x)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + a(t, x)u\varphi\right]d\Omega. \tag{5.85}$$

In the quality of $\mathcal{H}$ we take $L_2(\Omega)$. Then if we define the solution as usual, we get a equation of type (5.67).

(3) Let $\mathcal{H}$ be a Hilbert space. We consider the equation

$$\dot{y} = -y|y| + f(t), \tag{5.86}$$

where $y \in \mathcal{H}$ and $f \in C(\mathbb{R}, \mathcal{H})$. The next theorem takes place.

Theorem 5.7.3 (see [152]). *For any bounded on* $\mathbb{R}$ *function* $f \in C(\mathbb{R}, \mathcal{H})$ (5.86) *has a unique bounded on* $\mathbb{R}$ *solution* φ *and* $|\varphi(t)| \le \sqrt{2\|f\|}$ *for all* $t \in \mathbb{R}$*, where* $\|f\| = \sup\{|f(t)| : t \in \mathbb{R}\}$.

Corollary 5.38. *For any bounded on* $\mathbb{R}_+$ *function* $f \in C(\mathbb{R}, \mathcal{H})$ (5.86) *has at least one bounded on* $\mathbb{R}_+$ *solution.*

Proof. The formulated statement follows from Theorem 5.7.3. In fact, if $F \in C(\mathbb{R}, \mathcal{H})$ is bounded on $\mathbb{R}_+$, then the function $f \in C(\mathbb{R}, \mathcal{H})$, equal to $F(t)$ as $t \in \mathbb{R}_+$ and $F(0)$ as $t \in \mathbb{R}_-$, is bounded on $\mathbb{R}$ and according to Theorem 5.7.1, (5.86) has a unique bounded on $\mathbb{R}$ solution φ. The restriction of the function φ on $\mathbb{R}_+$ is the desired solution of (5.86). □

Lemma 5.39. *If the function $f \in C(\mathbb{R}, \mathcal{H})$ is bounded on $\mathbb{R}_+$, then all solutions of* (5.86) *are bounded on $\mathbb{R}_+$.*

Proof. Let $\varphi(t, x, f)$ be a solution of (5.86) passing through the point x as $t = 0$. Then according to [152, Lemma 1]

$$|\varphi(t, x_1, f) - \varphi(t, x_2, f)| \leq \frac{2|x_1 - x_2|}{2 + |x_1 - x_2|t} \tag{5.87}$$

for all $t \in \mathbb{R}_+$ and $x_1, x_2 \in \mathcal{H}$. Hence,

$$\lim_{t \to +\infty} |\varphi(t, x_1, f) - \varphi(t, x_2, f)| = 0 \tag{5.88}$$

for all $x_1, x_2 \in H$. Now to complete the proof of the lemma it is enough to refer to Corollary 5.38. □

Lemma 5.40. *For any asymptotically almost periodic function $f \in C(\mathbb{R}, \mathcal{H})$* (5.86) *has at least one asymptotically almost periodic solution.*

Proof. Let $f \in C(\mathbb{R}, \mathcal{H})$ be asymptotically almost periodic and

$$f(t) = p(t) + \omega(t) \tag{5.89}$$

for all $t \in \mathbb{R}$, where function $p \in C(\mathbb{R}, \mathcal{H})$ is almost periodic and $\lim_{t \to +\infty} |\omega(t)| = 0$. According to [152, Lemma 4] the equation

$$\frac{dx}{dt} = -x|x| + p(t) \tag{5.90}$$

has a unique almost periodic solution $q \in C(\mathbb{R}, \mathcal{H})$. Along with (5.89) we consider the equation

$$\frac{dx}{dt} = -x|x| + p(t) + \widetilde{\omega}(t), \tag{5.91}$$

where $\widetilde{\omega}(t) = \omega(t)$ for all $t \geq 0$ and $\widetilde{\omega}(t) = \omega(0)$ as $t < 0$. Denote by $\widetilde{\varphi}$ the unique bounded on $\mathbb{R}$ solution of (5.91). Let $\tau \geq 0$. Then $\widetilde{\varphi}^{(\tau)}(t) = \widetilde{\varphi}(t + \tau)$ is a unique bounded on $\mathbb{R}$ solution of the equation

$$\frac{dy}{dt} = -y|y| + p^{\tau}(t) + \widetilde{\omega}^{\tau}(t). \tag{5.92}$$

According to Theorem 5.7.3

$$|\overline{\varphi}^{\tau}(t) - q^{\tau}(t)| \leq \sqrt{2||\widetilde{\omega}^{\tau}||}. \tag{5.93}$$

Note that $\widetilde{\omega}^{(\tau)}(t) = \omega^{\tau}(t)$ for all $t \geq 0$ and $\widetilde{\omega}^{\tau}(t) = \omega^{\tau}(0)$ as $t < 0$ and, consequently,

$$\lim_{\tau \to +\infty} ||\widetilde{\omega}^{\tau}|| = 0. \tag{5.94}$$

From (5.93) and (5.94) it follows that $\lim_{t\to+\infty} |\widetilde{\varphi}(t) - q(t)| = 0$. Now to finish the proof of the lemma it is enough to note that the restriction of the function $\widetilde{\varphi}$ on $\mathbb{R}_+$ is asymptotically almost periodic solution of (5.86). □

Corollary 5.41. *For any asymptotically almost periodic function $f \in C(\mathbb{R}, \mathcal{H})$ all solutions of* (5.86) *are asymptotically almost periodic.*

Proof. The formulated statement follows from Lemma 5.39 and equality (5.88). □

Theorem 5.7.4. *If the mapping $f \in C(\mathbb{R}, \mathcal{H})$ is asymptotically almost periodic, then* (5.86) *is convergent.*

Proof. Let $Y := H^+(f) = \overline{\{f^{(\tau)} \mid \tau \in \mathbb{R}_+\}}$ (by bar it is denoted the closure in $C(\mathbb{R}, H)$) and $(Y, \mathbb{R}_+, \sigma)$ be a dynamical system of shifts on $H^+(f)$. Put $X := \mathcal{H} \times Y$ and define on X a dynamical system $(X, \mathbb{R}_+, \pi)$ by the following rule: $\pi(\tau, (x, g)) = (\varphi(t, x, g), g^{(\tau)})$, where $\varphi(t, x, g)$ is a solution of the equation

$$\frac{du}{dt} = -u|u| + g(t) \tag{5.95}$$

satisfying the initial condition $\varphi(0, x, g) = x$. Assume $h := pr_2 : X \to Y$ and consider the nonautonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h \rangle$. Let us show that the constructed nonautonomous dynamical system is convergent.

First of all, let us show that the system $(X, \mathbb{R}_+, \pi)$ is compactly dissipative. According to Lemma 5.39 the system $(X, \mathbb{R}_+, \pi)$ is point dissipative, since $\omega_x = \omega_{(p,q)} = H(p, q)$ for any $x \in X$ and, consequently, $\Omega_X = H(p, q)$ is compact.

Let $K \subset X$ be an arbitrary compact set and $\Sigma_K^+ := \{\pi^t x \mid x \in K,\ t \in \mathbb{R}_+\}$. Let us show that Σ_K^+ is relatively compact. Let $\{\overline{x}_n\} \subset \Sigma_K^+$. Then there exist $\{x_n\} \subset K$ and $\{t_n\} \subseteq \mathbb{R}_+$ such that $\overline{x}_n = \pi(x_n, t_n)$. Let $x_n := (u_n, g_n) \in \mathcal{H} \times H^+(f)$. Since K is a compact set, then the sequences $\{u_n\}$ and $\{g_n\}$ can be considered convergent. Assume $u := \lim_{n\to+\infty} u_n$ and $g := \lim_{n\to+\infty} g_n$. By the asymptotical almost periodicity of f we have

$$\lim_{n\to+\infty} \sup_{t\geq 0} |g_n(t) - g(t)| = 0. \tag{5.96}$$

Since $g \in H^+(f)$, the solution $\varphi(t, u, g)$ of (5.95) is asymptotically almost periodic and, hence, the sequence $\{\varphi(t_n, u, g)\}$ can be considered convergent. Let $\overline{u} := \lim_{n\to+\infty} \varphi(t_n, u, g)$. We will show that $\overline{x}_n \to \overline{x} = (\overline{u}, g)$. For this aim we note that

$$\begin{aligned} &|\varphi(t_n, u_n, g_n) - \varphi(t_n, u, g)| \\ &\quad \leq |\varphi(t_n, u_n, g_n) - \varphi(t_n, u, g_n)| + |\varphi(t, u, g_n) - \varphi(t, u, g)|. \end{aligned} \tag{5.97}$$

Put $w_n(t) := |\varphi(t,u,g_n) - \varphi(t,u,g)|$ and $\delta_n := \sup\{|g_n(t) - g(t)| : t \in \mathbb{R}_+\}$. According to [152, page 73]

$$\frac{dw_n(t)}{dt} \leq -\frac{1}{2}w_n^2(t) + \delta_n, \tag{5.98}$$

and taking into consideration that $w_n(0) = 0$, we obtain

$$w_n(t) \leq \sqrt{2\delta_n} \tag{5.99}$$

for all $t \in \mathbb{R}_+$. From (5.87), (5.96)–(5.99) it follows that

$$\lim_{n\to+\infty} |\varphi(t_n,u_n,g_n) - \varphi(t_n,u,g)| = 0, \tag{5.100}$$

and, consequently, $\overline{x}_n = (\varphi(t_n,u_n,g_n),g_n) \to (\overline{u},g) = \overline{x}$. So, Σ_K^+ is relatively compact. Assume $\mathbf{M} := H^+(K) = \overline{\Sigma_K^+}$ and

$$J := \Omega(\mathbf{M}) = \cap_{t\geq 0}\overline{\cup_{\tau\geq t}\pi^\tau\mathbf{M}}. \tag{5.101}$$

According to [112] the set J is compact and invariant. From Theorem 5.7.3 and Lemma 5.39 it follows that the unique compact invariant set of the dynamical system $(X,\mathbb{R}_+,\pi)$ is the set $\Omega_X = H(p,q)$. So, $\Omega(\mathbf{M}) = J = \Omega(X)$ and, hence, $(X,\mathbb{R}_+,\pi)$ is compactly dissipative dynamical system and its Levinson center $J_X = \Omega_X$. Now to finish the proof of the theorem it is sufficient to note that by Theorem 5.7.3 $J_X \cap X_y$ contains at most one point for any $y \in \omega_f = J_Y$. The theorem is proved. □

(4) Let $\mathcal{H}$ be a Hilbert space. In this point we will consider the equation

$$\frac{dx}{dt} = f(t,x), \tag{5.102}$$

where $f \in C(\mathbb{R}\times\mathcal{H},\mathcal{H})$ satisfies the condition

$$\operatorname{Re}\langle x_1 - x_2, f(t,x_1) - f(t,x_2)\rangle \leq -\kappa|x_1 - x_2|^\alpha \tag{5.103}$$

for all $t \in \mathbb{R}_+$ and $x \in \mathcal{H}$ ($\kappa > 0$ and $\alpha > 2$). Along with (5.102) we consider the family of equations

$$\frac{dy}{dt} = g(t,y), \quad (g \in H(f)), \tag{5.104}$$

where $H(f) := \overline{\{f^{(\tau)} \mid \tau \in \mathbb{R}\}}$, where $f^{(\tau)}$ is the shift of f onto τ with respect to the variable t and by bar it is denoted the closure in $C(\mathbb{R}\times\mathcal{H},\mathcal{H})$. Note that along with the function f any function $g \in H(f)$ satisfies condition (5.103) with the same constants κ and α. According to the results of [153, Chapter 2], if the function $f \in C(\mathbb{R}\times\mathcal{H},\mathcal{H})$ satisfies condition (5.103), then for every $u \in \mathcal{H}$ and $g \in H(f)$ (5.104) has a unique solution $\varphi(t,u,g)$ defined on $\mathbb{R}_+$ and passing through the point $u \in \mathcal{H}$ as $t = 0$; besides, the mapping $\varphi : \mathbb{R}_+ \times \mathcal{H} \times H(f) \to \mathcal{H}$ is continuous. Put now $Y := H(f)$ and by $(Y,\mathbb{R},\sigma)$ denote a dynamical system of shifts on $H(f)$. Further, let $X := \mathcal{H} \times H(f)$ and $\pi : \mathbb{R}_+ \times X \to X$ be the mapping defined by the equality $\pi(t,(u,g)) = (\varphi(t,u,g),g^{(t)})$. Then $(X,\mathbb{R}_+,\pi)$ is a semigroup dynamical system. At last, assume $h := pr_2 : X \to Y$. Then $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ is a nonautonomous dynamical system generated by (5.102).

Definition 5.42. As earlier, (5.102) is called convergent if the nonautonomous dynamical system $\langle (X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ generated by (5.102) is convergent.

In Chapter 3 we established (see Theorem 3.8.5 and Corollary 3.118) that (5.102) is convergent, if the right-hand side f is asymptotically almost periodic with respect to t and satisfies the condition (5.103) with the parameter $\alpha = 2$. Below we will establish the convergence of (5.102), when f satisfies condition (5.103) with the parameter $\alpha > 2$. Previously, let us give two auxiliary lemmas.

Lemma 5.43. *Let $f \in C(\mathbb{R}\times\mathcal{H},\mathcal{H})$ be such that the set $\{f^{(\tau)} \mid \tau \in \mathbb{R}\}$ is relatively compact in $C(\mathbb{R}\times\mathcal{H},\mathcal{H})$ and condition* (5.103) *is held. Then:*

(1) *for any $u \in \mathcal{H}$ the solution $\varphi(t,u,f)$ of* (5.102) *is compact on $\mathbb{R}_+$ (i.e., $\varphi(\mathbb{R}_+,u,f)$ is a relatively compact set in $\mathcal{H}$);*
(2) *for all $t \in \mathbb{R}_+$ and $x_1,x_2 \in \mathcal{H}$*

$$\begin{aligned}|\varphi(t,x_1,f)-\varphi(t,x_2,f)| &\le \left(|x_1-x_2|^{2-\alpha}+(\alpha-2)t\right)^{1/(2-\alpha)}\\ &= |x_1-x_2|\left(1+|x_1-x_2|^{\alpha-2}(\alpha-2)t\right)^{1/(2-\alpha)}.\end{aligned} \tag{5.105}$$

Proof. Let us define a function $F \in C(\mathbb{R}\times\mathcal{H},\mathcal{H})$ by the following rule

$$F(t,x) := \begin{cases} f(t,x), & \text{for } (t,x) \in \mathbb{R}_+\times\mathcal{H},\\ f(0,x), & \text{for } (t,x) \in \mathbb{R}_-\times\mathcal{H}.\end{cases} \tag{5.106}$$

It is easy to check that the function F possesses the next properties:

(a) $\{F^{(\tau)} \mid \tau \in \mathbb{R}\}$ is relatively compact in $C(\mathbb{R}\times\mathcal{H},\mathcal{H})$;
(b) $\operatorname{Re}\langle x_1-x_2, F(t,x_1)-F(t,x_2)\rangle \le -\kappa|x_1-x_2|^{\alpha}$ for all $t \in \mathbb{R}$ and $x_1,x_2 \in \mathcal{H}$.

According to [153, Theorem 2.2.3.1] the equation

$$\frac{dx}{dt} = F(t,x) \tag{5.107}$$

has a unique compact on $\mathbb{R}$ solution $\varphi(t,x_0,F)$ and for every two solutions $\varphi(t,x_1,F)$ and $\varphi(t,x_2,F)$ there takes place the inequality

$$|\varphi(t,x_1,F)-\varphi(t,x_2,F)| \le \left(|x_1-x_2|^{2-\alpha}+(2-\alpha)t\right)^{1/(2-\alpha)} \tag{5.108}$$

for all $t \in \mathbb{R}_+$, $x_1,x_2 \in \mathcal{H}$ and, consequently, $\lim_{t\to+\infty}|\varphi(t,x_1,F)-\varphi(t,x_0,F)| = 0$ for all $x \in \mathcal{H}$. The last relation imply that all solutions of (5.107) are compact on $\mathbb{R}_+$. Now to complete the proof of the lema it is enough to note that $\varphi(t,x,f) = \varphi(t,x,F)$ for all $t \in \mathbb{R}_+$. The lemma is proved. □

Lemma 5.44. *Let α, κ and ε be positive numbers. Then on $\mathbb{R}_+$ the scalar equation*

$$\frac{dx}{dt} = -\kappa x^{\alpha} + \varepsilon \tag{5.109}$$

defines a semiflow $\varphi_{\varepsilon}(x,t)$ which has a unique stationary point $x_{\varepsilon} = (\varepsilon/\kappa)^{1/\alpha}$ and

$$0 \le \varphi_{\varepsilon}(x,t) \le x_{\varepsilon} \tag{5.110}$$

for all $x \in [0, x_{\varepsilon}]$ and $t \in \mathbb{R}_+$.

Proof. The proof is obvious. □

Theorem 5.7.5. *Let $f \in C(\mathbb{R} \times \mathcal{H}, \mathcal{H})$ be asymptotically almost periodic with respect to $t \in \mathbb{R}$ uniformly with respect to x on compact subsets from $\mathcal{H}$ and satisfy condition* (5.103). *Then* (5.102) *is convergent.*

Proof. Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h \rangle$ be a nonautonomous dynamical system generated by (5.102). Since $Y = H^+(f)$ and f is asymptotically almost periodic, then $(Y, \mathbb{R}_+, \sigma)$ is compactly dissipative an $J_Y = \omega_f$.

Let us show that for every compact subset $K \subset \mathcal{H}$ the set $\varphi(\mathbb{R}_+, K, H^+(f)) = \{\varphi(t,x,g) \mid t \in \mathbb{R}_+,\ x \in K,\ g \in H^+(f)\}$ is relatively compact. Let $\{y_n\} \subseteq \varphi(\mathbb{R}_+, K, H^+(f))$. Then there exist $\{t_n\} \subset \mathbb{R}_+$, $\{x_n\} \subseteq K$ and $\{g_n\} \subseteq H^+(f)$ such that $y_n = \varphi(t_n, x_n, g_n)$. In virtue of the compactness of K and $H^+(f)$ the sequences $\{x_n\}$ and $\{g_n\}$ can be considered convergent. Assume $x := \lim_{n\to+\infty} x_n$ and $g := \lim_{n\to+\infty} g_n$. If the sequence $\{t_n\}$ is bounded, then the sequence $\{\varphi(t_n, x_n, g_n)\}$ is relatively compact and the necessary statement is proved. Let now $\{t_n\}$ be not bounded. Then without loss of generality we can consider that $t_n \to +\infty$ as $n \to +\infty$. Since according to Lemma 5.43 the set $\varphi(\mathbb{R}_+, x, g)$ is relatively compact, the sequence $\{\varphi(t_n, x, g)\}$ can be considered convergent. Put $\bar{x} := \lim_{n\to+\infty} \varphi(t_n, x, g)$ and show that $\varphi(t_n, x_n, g_n) \to \bar{x}$ as $n \to +\infty$. For this aim we note that

$$\begin{aligned} &|\varphi(t_n, x_n, g_n) - \varphi(t_n, x, g)| \\ &\quad \le |\varphi(t_n, x_n, g_n) - \varphi(t_n, x, g_n)| + |\varphi(t_n, x, g_n) - \varphi(t_n, x, g)|. \end{aligned} \tag{5.111}$$

Let us estimate the terms of the right-hand side of (5.111). By (5.105)

$$|\varphi(t_n, x_n, g_n) - \varphi(t_n, x, g_n)| \le |x_n - x|. \tag{5.112}$$

On the other hand, if $\omega_n(t) := |\varphi(t,x,g_n) - \varphi(t,x,g)|^2$, then

$$\begin{aligned} \omega_n'(t) &= 2\,\mathrm{Re}\,\langle g_n(t, \varphi(t,x,g_n)) - g(t, \varphi(t,x,g)), \varphi(t,x,g_n) - \varphi(t,x,g)\rangle \\ &\le 2\,\mathrm{Re}\,\langle g_n(t, \varphi(t,x,g_n)) - g_n(t, \varphi(t,x,g)), \varphi(t,x,g_n) - \varphi(t,x,g)\rangle \\ &\quad + 2\,\mathrm{Re}\,\langle g_n(t, \varphi(t,x,g)) - g(t, \varphi(t,x,g)), \varphi(t,x,g_n) - \varphi(t,x,g)\rangle \\ &\le -2\kappa |\varphi(t,x,g_n) - \varphi(t,x,g)|^{\alpha} + 2\varepsilon_n |\varphi(t,x,g_n) - \varphi(t,x,g)| \end{aligned} \tag{5.113}$$

for all $t \in \mathbb{R}_+$, where

$$\varepsilon_n = \sup\{|g_n(t,x) - g(t,x)| : x \in \overline{\varphi(\mathbb{R}_+, x, g)}, t \in \mathbb{R}_+\}. \tag{5.114}$$

By asymptotical almost periodicity of $f \in C(\mathbb{R} \times \mathcal{H}, \mathcal{H})$ and compactness on $\mathbb{R}_+$ of the solution $\varphi(t,x,g)$, we have $\varepsilon_n \to 0$ as $n \to +\infty$. According to [153, Theorem 1.1.1.2]

$$\omega_n(t) \leq \psi_n(t,0), \tag{5.115}$$

where $\psi_n(t,x)$ is the solution of the equation

$$\frac{du}{dt} = -2\kappa u^{\alpha/2} + 2\varepsilon_n u^{1/2} \tag{5.116}$$

passing through the point x as $t = 0$. Assume $y := u^{1/2}$. Then from (5.116) we get

$$\frac{dy}{dt} = -\kappa y^{\alpha} + \varepsilon_n. \tag{5.117}$$

So, $\sqrt{\psi_n(t,x)}$ is a solution of (5.117) and by Lemma 5.43 $\sqrt{\psi_n(t,0)} \leq \sqrt{x_{\varepsilon_n}} = (\varepsilon_n/\kappa)^{1/2\alpha}$ and, consequently,

$$|\varphi(t,x,g_n) - \varphi(t,x,g)| \leq \left(\frac{\varepsilon_n}{\kappa}\right)^{1/2\alpha} \tag{5.118}$$

for all $t \in \mathbb{R}_+$. From inequalities (5.111), (5.112), and (5.118) it follows that

$$\lim_{n\to+\infty} |\varphi(t_n,x_n,g_n) - \varphi(t_n,x,g)| = 0, \tag{5.119}$$

and hence $\{y_n\}$ is convergent and the necessary statement is proved.

Since along with the function $f \in C(\mathbb{R} \times \mathcal{H}, \mathcal{H})$ every function $g \in H^+(f)$ also satisfies condition (5.103), then according to Lemma 5.36,

$$|\varphi(t,u_1,g) - \varphi(t,u_2,g)| \leq \frac{|u_1 - u_2|}{\left(1 + |u_1 - u_2|^{\alpha-2}(\alpha-2)t\right)^{1/(\alpha-2)}} \tag{5.120}$$

for all $u_1, u_2 \in \mathcal{H}$ and $g \in H^+(f)$. Now to finish the proof of the theorem it is necessary to refer to Theorem 2.6.1. □

Theorem 5.7.6. *Let $f \in C(\mathbb{R} \times \mathcal{H}, \mathcal{H})$ be asymptotically recurrent with respect to $t \in \mathbb{R}$ uniformly with respect to x on compacts from $\mathcal{H}$, the space $\mathcal{H}$ be finite-dimensional and the function f satisfy condition* (5.103) *with the parameter $\alpha > 2$. Then* (5.102) *is convergent.*

Proof. First of all, let us show that the nonautonomous dynamical system $\langle (X,\mathbb{R}_+,\pi), (Y,\mathbb{R}_+,\sigma), h\rangle$ generated by (5.102) (see Example 3.1 and Corollary 3.2) is dissipative.

Assume $m := \sup\{|f(t,0)| : t \in \mathbb{R}_+\}$ and $w(t) := |\varphi(t,u,g)|^2$, where $u \in \mathcal{H}$ and $g \in H^+(f)$. Then

$$\begin{aligned} w'(t) &= 2\,\mathrm{Re}\,\langle g(t,\varphi(t,u,g)),\varphi(t,u,g)\rangle \\ &= 2\,\mathrm{Re}\,\langle g(t,\varphi(t,u,g)) - g(t,0),\varphi(t,u,g)\rangle + 2\,\mathrm{Re}\,\langle g(t,0),\varphi(t,u,g)\rangle \\ &\le -2\kappa w^{\alpha/2}(t) + 2mw^{1/2}(t). \end{aligned} \tag{5.121}$$

According to [153, Theorem 1.1.1.2]

$$|\varphi(t,u,g)|^2 \le \psi(t,|u|^2) \tag{5.122}$$

for all $t \in \mathbb{R}_+$, where $\psi(t,x)$ $(x \ge 0)$ is the solution of the equation

$$\frac{dx}{dt} = -2\kappa x^{\alpha/2} + 2mx^{1/2} \tag{5.123}$$

passing through the point x as $t = 0$. It is easy to see that

$$\lim_{t\to+\infty} |\psi(t,x)| \le \left(\frac{m}{\kappa}\right)^{2/(\alpha-1)} \tag{5.124}$$

for all $x \ge 0$. From (5.122) and (5.124) it follows the necessary statement.

Since $Y = H^+(f)$, then $(Y,\mathbb{R}_+,\sigma)$ is compactly dissipative and $J_Y = \omega_f$. Further, by the finite-dimensionality of the space $\mathcal{H}$ the dynamical system $(X,\mathbb{R}_+,\pi)$ is compactly dissipative too. Denote by J_X its Levinson center and show that for any $y \in J_X$ the set $J_X \cap X_y$ contains at most one point. Put $V(x_1,x_2) := |x_1 - x_2|^2$. Then

$$\begin{aligned} V(\varphi(t,u_1,g),\varphi(t,u_2,g)) &= |\varphi(t,u_1,g) - \varphi(t,u_2,g)|^2, \\ \frac{dV(\varphi(t,u_1,g),\varphi(t,u_2,g))}{dt} & \\ &= 2\,\mathrm{Re}\,\langle g(t,\varphi(t,u_1,g)) - g(t,\varphi(t,u_2,g)),\varphi(t,u_1,g) - \varphi(t,u_2,g)\rangle \\ &\le -2\kappa|\varphi(t,u_1,g) - \varphi(t,u_2,g)|^\alpha = -2\kappa V^{\alpha/2}(\varphi(t,u_1,g),\varphi(t,u_2,g)) \end{aligned} \tag{5.125}$$

for all $t \in \mathbb{R}_+$. From (5.125) it follows that

$$|\varphi(t,u_1,g) - \varphi(t,u_2,g)| < |u_1 - u_2| \tag{5.126}$$

for all $t > 0$ and $u_1,u_2 \in \mathcal{H}$ $(u_1 \ne u_2)$. Let now $g \in \omega_f = J_Y$ and $(u_1,g),(u_2,g) \in J_X \cap X_g$. Then by [113, Theorem 1], the solutions $\varphi(t,u_1,g)$ and $\varphi(t,u_2,g)$ are jointly recurrent and if $u_1 \ne u_2$, then there takes place (5.126) and it contradicts to the recurrence of $|\varphi(t,u_1,g) - \varphi(t,u_2,g)|$. The obtained contradiction proves that $J_X \cap X_y$ contains no more than one point for any $y \in J_Y$. The theorem is proved. □

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